The Method of Fundamental Solutions and a Taste of Python

Informal Numerical Analysis Seminar

University of Bath

Friday 11 April 2008

Introduction

- BICS talk by Jon Trevelyan last Monday (7 April)
- About method for solving the Helmholtz equation
- very inspiring
- got me thinking

Summary of talk by Jon Trevelyan

- **Helmholtz equation**
- **boundary integral** equation formulation
- standard finite elements (**FE**) for function on boundary
- boundary element method **BEM**
- **enrich** with plane wave expansion
- **decompose** solution as
- in general write solution as linear combination of **basis** functions
- **collocation** of the boundary integral equation
- gives rise to **system** for unknowns on the boundary
- curve (1D) for 2D, surface (2D) for 3D

- Dirichlet/Neumann
- building **matrix**, involves solving **integrals**
- his approach subdivide integration domain, Gauss along wave crests,
- rule for oscillatory integrals (steepest descent) in direction of wave
- 1D integral for 2D, 2D integral for 3D

Basis functions

- idea: can we choose **beter basis** functions?
- sidestep
- standard finite element (**FE**) method (e.g. for elliptic equations)
- Dirichlet problem
- choose basis functions that satisfy boundary conditions (**BC**) (but don't solve PDE)
- e.g. triangulation and **hat** functions for nodes in interior
- then do Galerkin or collocation inside domain
- homogeneous **Helmholtz** equation
- basis functions that **solve PDE**? (but don't satisfy boundary conditions)
- for homogeneous Helmholtz (Laplace) this is possible
- 2D Hankel function $H_0^{(1)}(k r)$
- **spherical** wave 3D $\exp(i k r)/r$
- point source in 3D
- **cylindrical** wave, line source in 3D
- essentially point source in 2D
- **fundamental** solutions
- **approximate** solution solves **PDE** in domain
- need to find **coefficients** such that boundary conditions (**BC**) satisfied
- **collocation**
- very similar in flavour to boundary element method (**BEM**)
- but **no integration** necessary

Detailed explanation of method (see notes)

Further discussion

- of course I wasn't the first to come up with this
- some searching
- memory of talk at **Dundee**
- method of **particular solutions** for PDE eigenvalue problems
- paper by **Trefethen**
- work by **Moler**, **Matlab logo**
- **Method of Fundamental Solutions**
- paper by **Barnett and Betcke**
- authors used **Matlab**
- they make the **link** with the boundary integral formulation (**BEM**)
- **analysis** for interior Helmholtz for disc
- convergence, conditioning, stability
- conditioning of system, size of coefficients
- solution as analytic function
- e.g. point source outside of domain
- best to choose points "inbetween" singularity and domain
- choice of curves

Advantages

- **interior** or **exterior** problems
- automatically satisfies boundary conditions at **infinity** (**Sommerfeld** radiation conditions)
- very easy to **implement**
- works in **2D**, **3D**, ...
- **meshless**

Disadvantages

- **conditioning**
- but B&B article shows that this can be alleviated by
- choosing **source locations** well
- also problem for other methods

Implementation

- How hard is this to implement?
- walk through **python** code
- show that we can very easily get very **similar** results to **Matlab**
- many **libraries** available
- here we use **numpy**, **scipy** and **matplotlib**
- module/file doc string
- importing modules
- numpy, scipy, pylab (matlab-like plotting interface to matplotlib)
- importing specific functions, submodules (, variables, classes)
- defining new functions
- scipy documentation
- tutorials, some reference documentation, api docs, wiki, source
- here api docs were useful
- scipy.special Hankel, derivative of Hankel, Bessel
- unit_circle, circle (only points and gradients needed, meshless)
- fundamental solutions
- source using Bessel function of the second kind (\(Y_0\))
- helmholtz class
- example: unit disc, bc based on Bessel point source outside
- Dirichlet/Neumann
- numerical values for parameters
- plots using matplotlib, very similar to Matlab

Demonstration

- Figures 4a, 4b
- Figures 3a, 3b, 3c
- other examples: discs with Dirichlet and Neumann conditions
- Animations

Idea for further exploration

- talk at Gene Golub memorial day by Godela Scherer
- about work with Gene Golub and Victor Pereyra on optimisation
- Gene Golub QR and SVD for LS
- generalised to separable nonlinear optimisation
- VARPRO/PORT
- paper illustrates that we want to avoid growth of coefficients
- minimise not just IIrII^2 but IIrII^2 + \nu IIcII^2
- standard regularisation
- Golub showed how to this for QR
- more stable for ill conditioned problems
- normal equations kappa^2, QR kappa
- same equations as before, but in addition n equations of the form \(\sqrt{\nu} c_j = 0\)