Multigrid Waveform Relaxation for Anisotropic Partial Differential Equations

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Overview

- ☞ Overview
- Model Problems
- Discretisation
- Waveform Relaxation
- Multigrid Waveform Relaxation
- Fourier-Laplace Analysis
- Results
- Conclusion

Time-Dependent Problems

- parabolic PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f$$

- elliptic PDE

$$\mathcal{L}u + f = 0$$

Model Problems

- Diffusion Equation

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Anisotropic Diffusion Equation

$$\mathcal{L} = \varepsilon\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Rotated Anisotropic Diffusion Equation

$$\mathcal{L} = (\varepsilon c^2 + s^2)\frac{\partial^2}{\partial x^2} + (c^2 + \varepsilon s^2)\frac{\partial^2}{\partial y^2} + 2(\varepsilon - 1)cs\frac{\partial^2}{\partial x \partial x}$$

$$c = \cos\beta,\ s = \sin\beta$$

Discretisation

- method of lines (spatial discretisation)
  - approximate by grid of continuous functions
  - finite difference approximation of operator
  - system of ODEs

$$\dot{u} = Lu + f$$

  - eg. anisotropic diffusion equation (stencil notation)

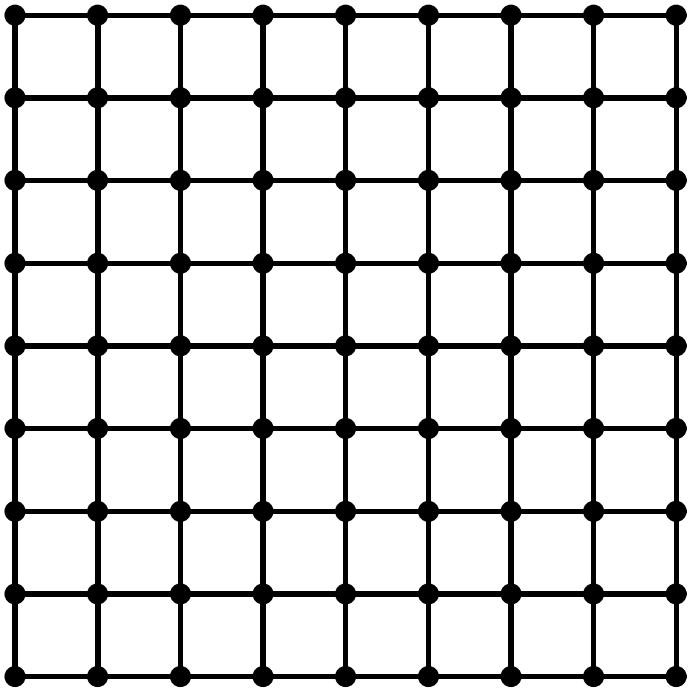
$$L = \begin{bmatrix} & \frac{1}{h_y^2} & \\ \frac{\varepsilon}{h_x^2} & -\frac{2\varepsilon}{h_x^2} - \frac{2}{h_y^2} & \frac{\varepsilon}{h_x^2} \\ & \frac{1}{h_y^2} & \end{bmatrix}$$

    rotated equation : 9-point stencil

- approximate solution of ODEs $\rightarrow$ time discretisation
  - approximate by grid of discrete functions
  - stiff problems $\rightarrow$ implicit solvers

Grid

- grid of continuous or discrete functions



Waveform Relaxation

- iteratively solve set of simpler subsystems
- choose splitting of operator L

  $$L = P + Q$$

  resulting in a simpler set of equations

  $$\dot{u}^{(\nu+1)} = Pu^{(\nu+1)} + Qu^{(\nu)} + f$$

- point relaxation : Jacobi, Gauss-Seidel
- line relaxation : horizontal and vertical line Jacobi, Gauss-Seidel, zebra Gauss-Seidel
- Jacobi

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu)} + f$$

- Lexicographic Gauss-Seidel

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & & \\ \bullet & \ddots & \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ & & \bullet \\ & & \bullet \end{bmatrix} u^{(\nu)} + f$$

- Red-Black Gauss-Seidel

$i + j$ even

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \\ & \ddots & \bullet \\ & & \end{bmatrix} u^{(\nu)} + f$$

$i + j$ odd

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \\ & \ddots & \bullet \\ & & \end{bmatrix} u^{(\nu+1)} + f$$

- Four Colour Gauss-Seidel

- Horizontal Zebra Gauss-Seidel

i even

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ & & \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu)} + f$$

i odd

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ & & \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + f$$

- Vertical Zebra Gauss-Seidel
- Alternating Zebra Gauss-Seidel

Multigrid

- system $\dot{u}_h = L_h u_h + f_h$
- principle
- algorithm

  pre smoothing solve
  $$\dot{v}_h^{(1)} = P_h v_h^{(1)} + Q_h v_h^{(0)} + f_h$$

  defect calculation
  $$d_h = L_h v_h^{(1)} + f_h - \dot{v}_h^{(1)} = Q_h (v_h^{(1)} - v_h^{(0)})$$

  residual equation solve recursively
  $$\dot{e}_H = L_h e_H + I_h^H d_h$$

  correction $v_h^{(2)} = v_h^{(1)} + I_H^h e_H$

  post smoothing solve
  $$\dot{v}_h^{(3)} = P_h v_h^{(3)} + Q_h v_h^{(2)} + f_h$$

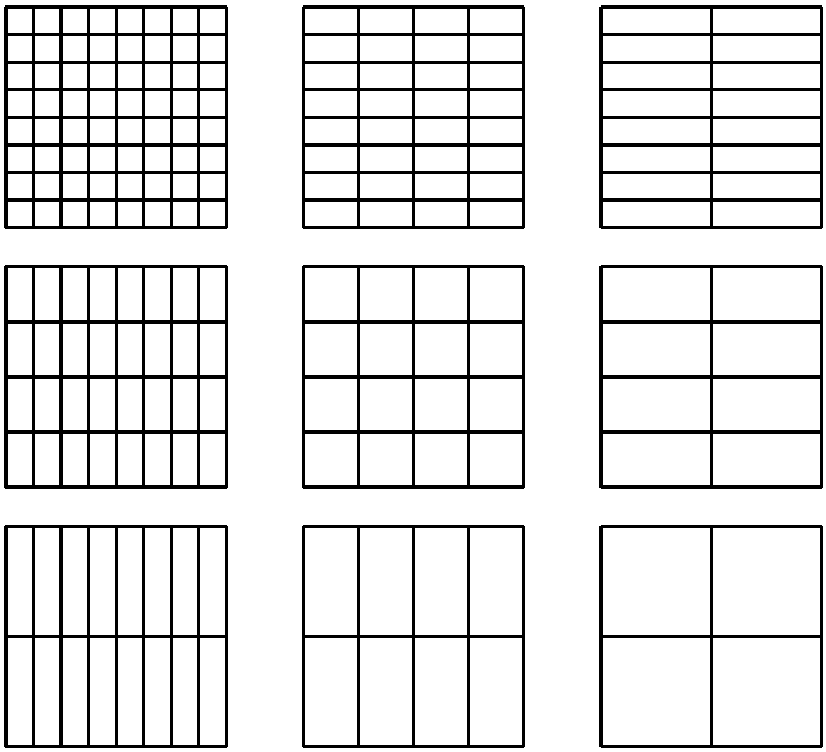
[Lubich and Ostermann, 1987; Vandewalle, 1993]

Smoothing

- standard waveform relaxation methods
- line relaxation for anisotropic problems

Coarsening

- standard coarsening : double grid spacing in both directions
- semicoarsening for anisotropic problems
  - x-semicoarsening, y-semicoarsening
  - alternating semicoarsening
    (MG-S [Washio and Oosterlee, 1995, 1998])



Fourier-Laplace Analysis

- error of successive approximations

$$e^{(\nu+1)} = \mathcal{M} e^{(\nu)}$$

- asymptotic convergence factor of $\mathcal{M}$ and its Laplace transform $M(z)$ are related as follows

$$\rho(\mathcal{M}) = \max_{z \in \Sigma} \rho(M(z))$$

- $M(z)$ is iteration operator of Laplace transform of the original equation

$$(zI - L)u = f$$

- spectral picture
- Σ depends on type of waveform relaxation
  - continuous or discrete
  - finite time, infinite time or time periodic

[Vandewalle; Janssen; Ta'asan and Zhang]

Local Mode Fourier Analysis

- Two Grid Analysis
  - Standard Coarsening

$$M_s = S^{\nu_1} K_s S^{\nu_2}$$

$$K_s = I - I_H^h L_H^{-1} I_h^H L_h$$

  - Semicoarsening

$$M_x = S^{\mu_1} K_x S^{\mu_2} \qquad M_y = S^{\mu_1} K_y S^{\mu_1}$$

  - Alternating Semicoarsening

$$M_a = (M_x M_y)^{\nu_1} K_s (M_x M_y)^{\nu_2}$$

- operator transforms exponential Fourier modes

$$\rho(M(z)) = \max_{\theta} \rho(M(z, \theta))$$

[Vandewalle; Horton; Ta'asan and Zhang]

[Wesseling; Oosterlee]

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Results

Model Problem

- Rotated Anisotropic Diffusion Equation (ε, β)

Special Cases

- Anisotropic Diffusion Equation (ε, $\beta = 0$)
- Diffusion Equation ($\varepsilon = 1$, $\beta = 0$)

Smoothers

- Red-Black or Four-Colour Gauss-Seidel
- Horizontal, Vertical and Alternating Zebra Gauss-Seidel

Coarsening

- Standard Coarsening
- X- and Y-Semicoarsening
- Alternating Semicoarsening

Correction for computational work

- convergence factor $\rho \quad (R = \log(\rho))$
- method requires α "work units"
- n iterations $\rightarrow \rho^n$, $n\alpha$
- convergence factor "per work unit"

$$\tilde{\rho} = \sqrt[\alpha]{\rho} \qquad \tilde{R} = \frac{R}{\alpha}$$

- correction factor α

Point Relaxation	1
Line Relaxation	2
Standard Coarsening (V)	$\frac{4}{3}$
Semicoarsening (V)	2
Alternating Semicoarsening (V)	$\frac{16}{3}$

(V) : V-cycle

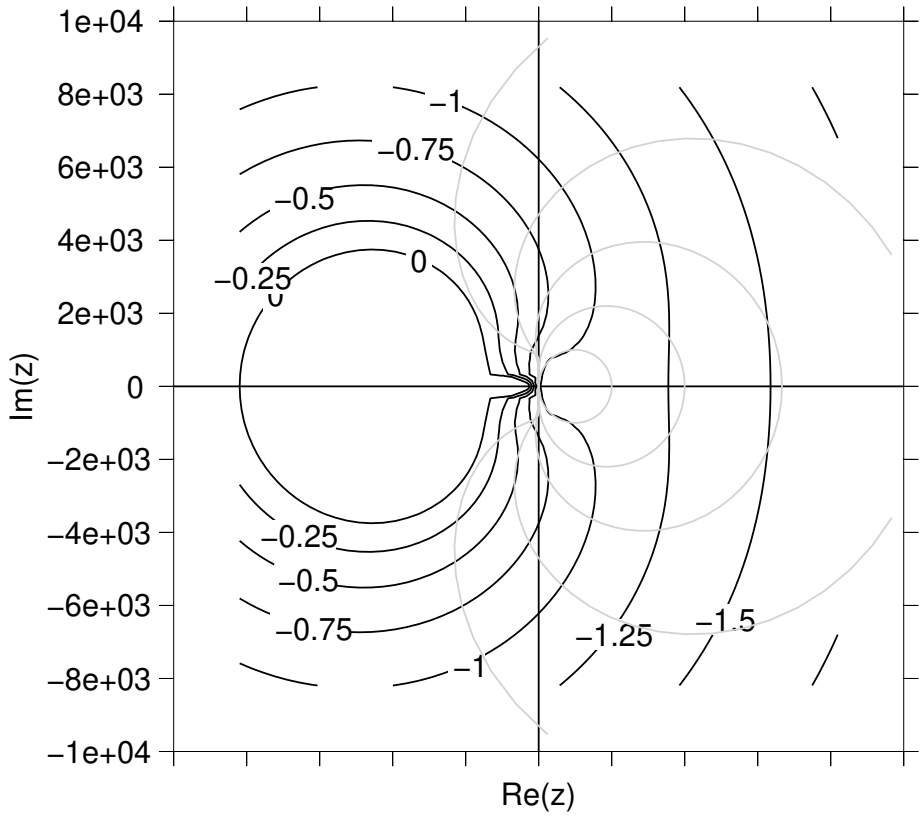
- eg. Four Colour smoother with Alternating Semicoarsening (V)

$$\alpha = 2 \times 1 \times \frac{16}{3} = 10\frac{1}{3}$$

Spectral Image

- diffusion equation ($\varepsilon = 1, \beta = 0$)
- red-black Gauss-Seidel
- grid spacing $h = 2^{-5}$
- time step $h_t = 10^{-3}$
- boundary loci for BDF1-5

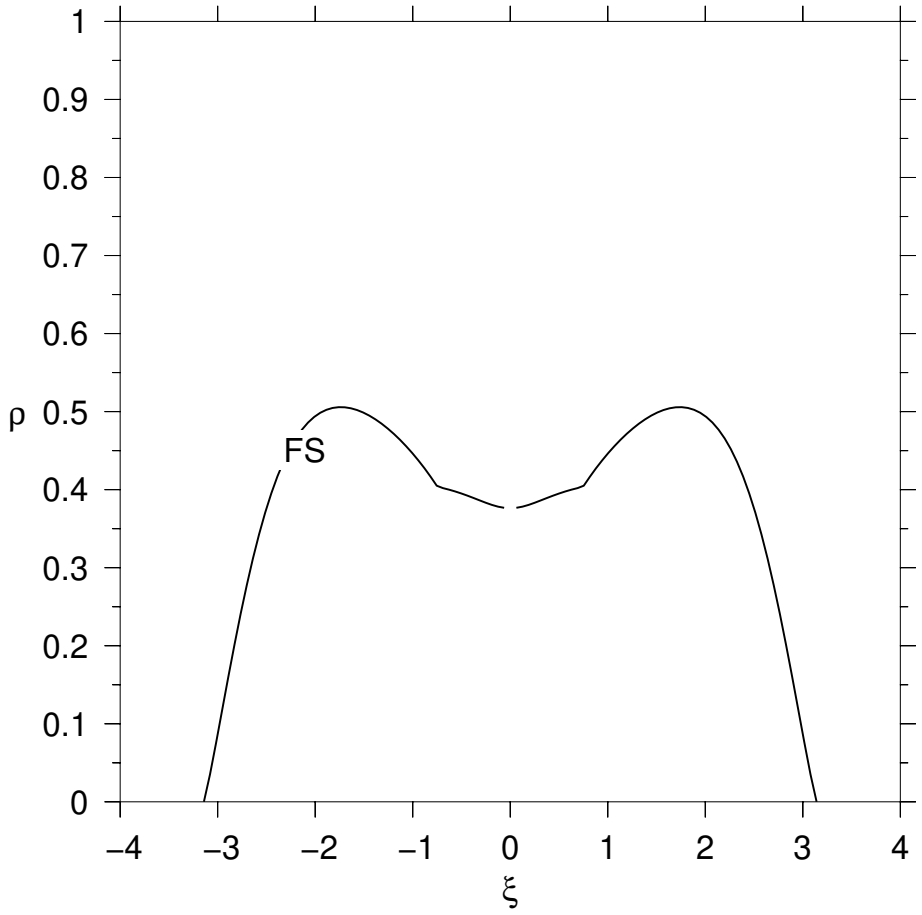
$$\rho(M(z))$$



$$h = 2^{-5}, \quad h_t = 10^{-3}, \quad \varepsilon = 1, \quad \beta = 0$$

$\rho(M(z))$ along the imaginary axis

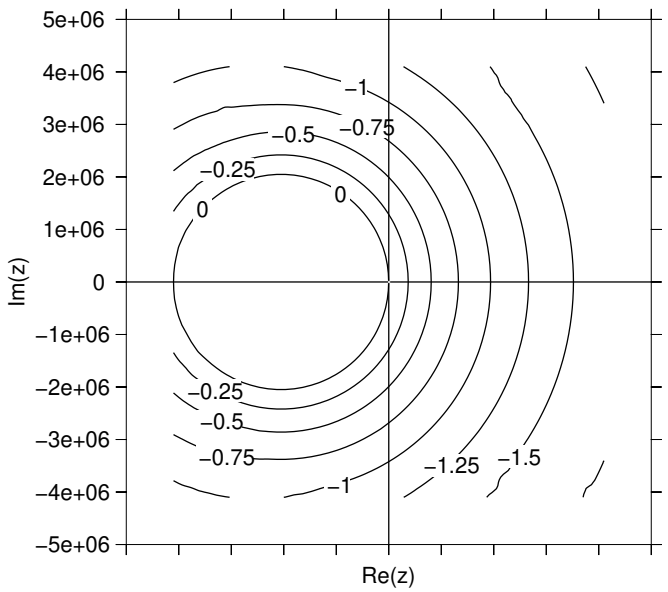
$$z = \tfrac{1}{2}\tfrac{1-w}{1+w}, \quad w = e^{i\xi}$$



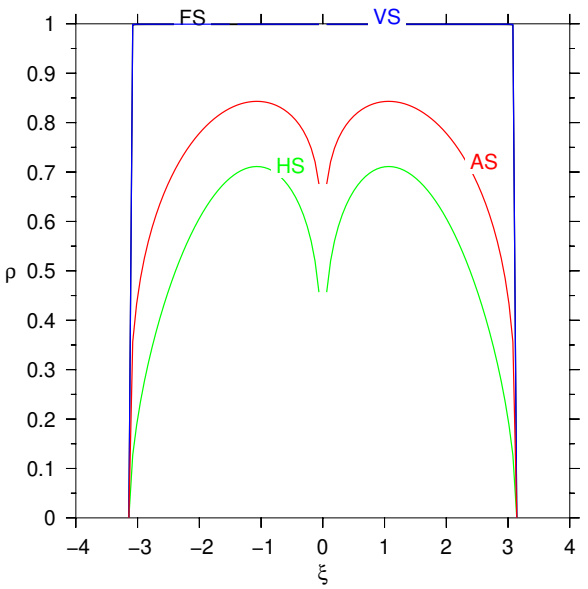
F :	red-black or four-colour	S :	standard coarsening
H :	horizontal zebra	X :	x-coarsening
V :	vertical zebra	Y :	y-coarsening
A :	alternating zebra	A :	alternating semicoarsening

$$h = 2^{-5}, \quad h_t = 10^{-3}, \quad \varepsilon = 10^3, \quad \beta = 0$$

$\rho(M(z))$



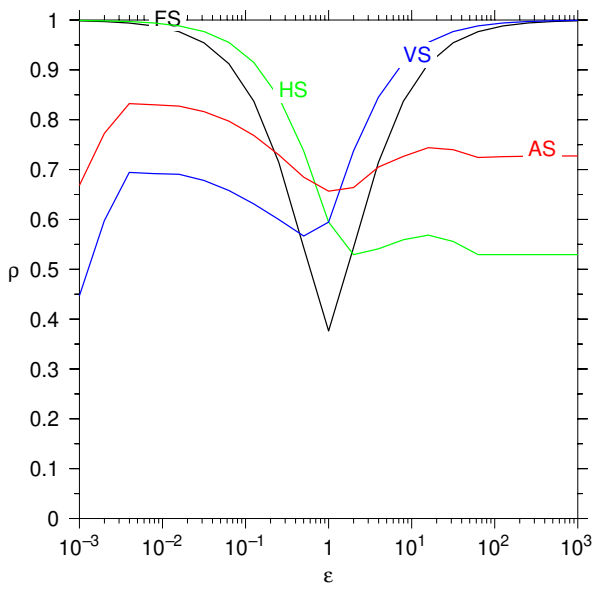
$\rho(M(z))$ along the imaginary axis



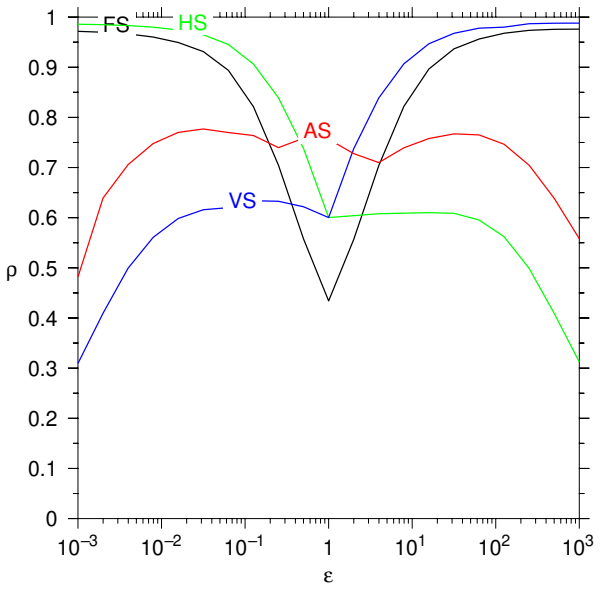
$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \beta = 0$$

$\rho(\varepsilon)$ for FS, HS, VS, AS

analysis



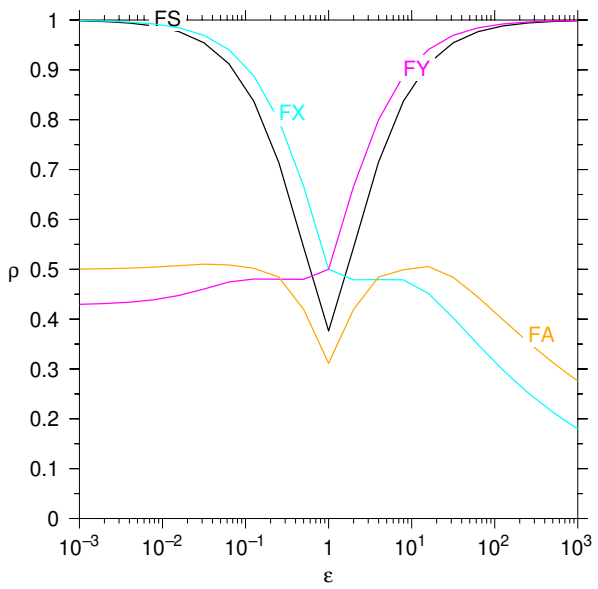
numerical



$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \beta = 0$$

$\rho(\varepsilon)$ for FS, FX, FY, FA

analysis



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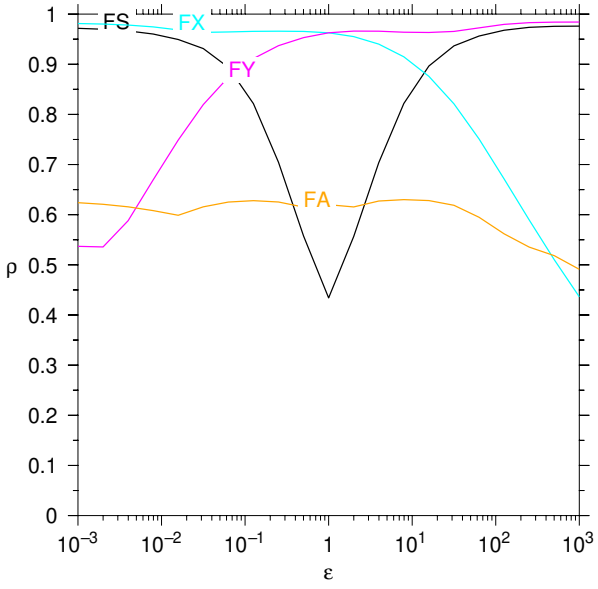
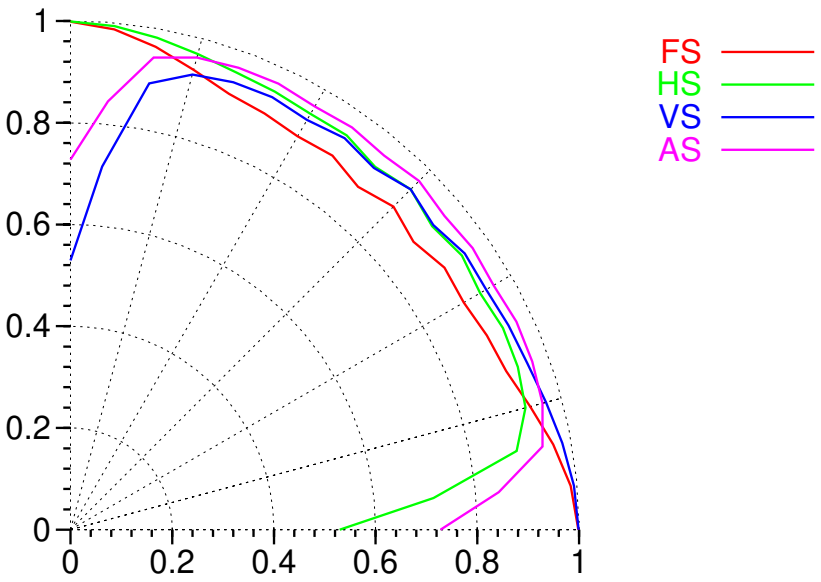


$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \varepsilon = 10^3$$

$\rho(\beta)$ for FS, HS, VS, AS

analysis



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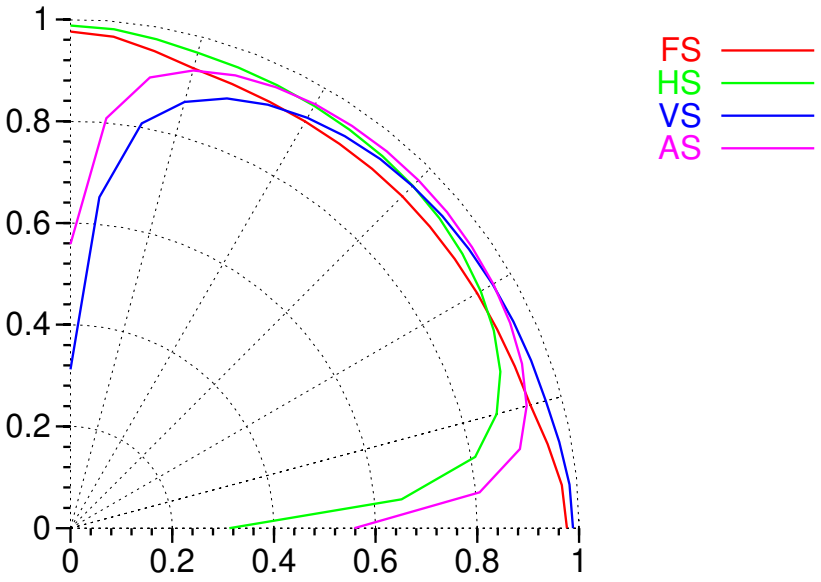

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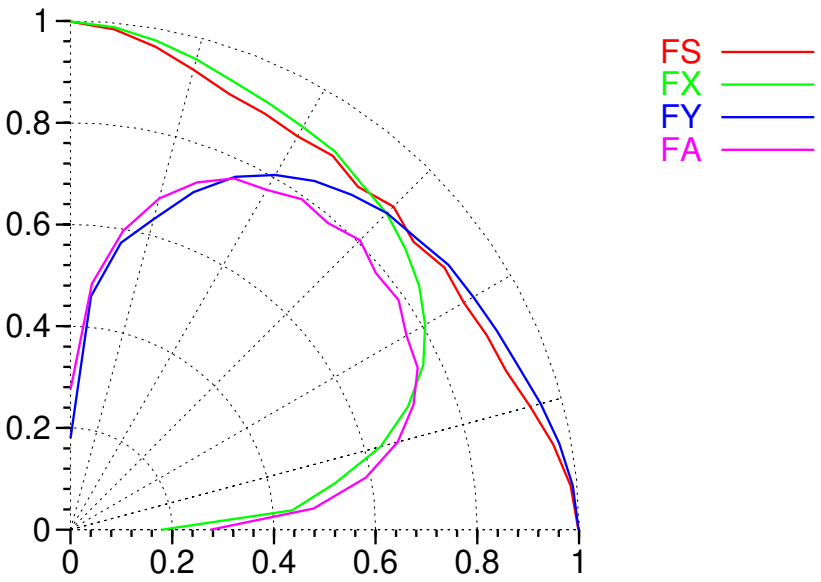

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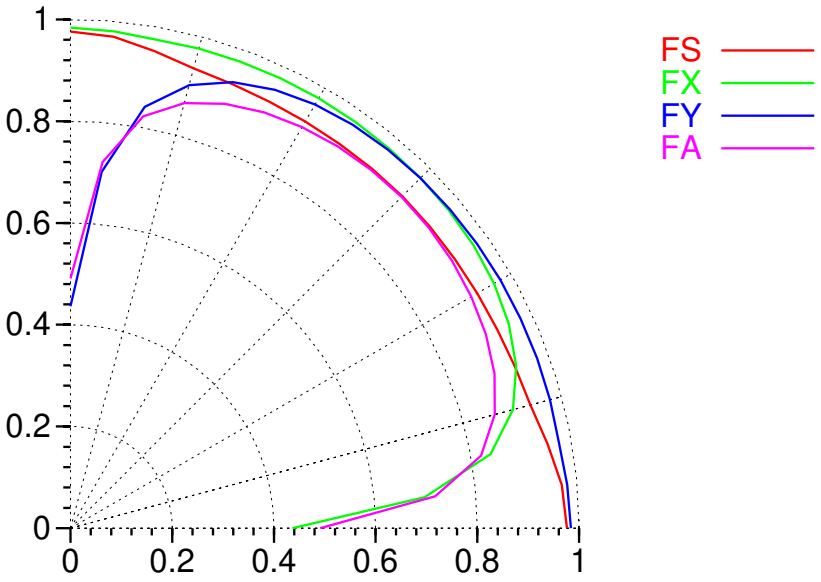

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Conclusion

- Standard waveform relaxation $\rightarrow$ slow convergence
- Multigrid waveform relaxation
  - $\varepsilon = 1, \beta = 0$ red-black
  - $\varepsilon \neq 1, \beta = 0$ line smoothers, semicoarsening
  - $\varepsilon \neq 1, \beta \neq 0$ alternating zebra or semicoarsening
- methods for elliptic problems can be adapted for parabolic problems

Future

- other test problems
- delay partial differential equations
- time-periodic solutions