BICS Away Day
Theme D Presentation

Chris Budd, Ivan Graham, Robert Scheichl,
Jan Van lent

Theme D

Tuesday 16 September 2008

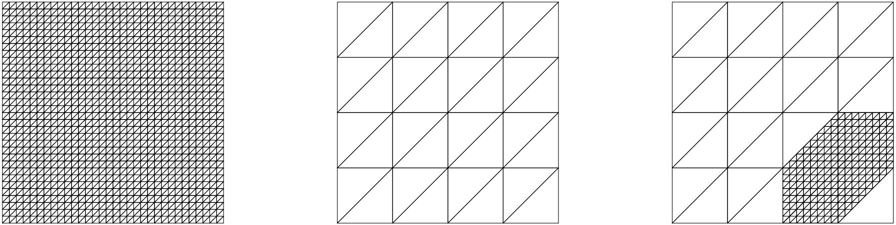
Domain Decomposition Methods for Elliptic PDEs

- typical equation : diffusion with variable coefficients

$$-\nabla \cdot (\alpha \nabla u) = f$$

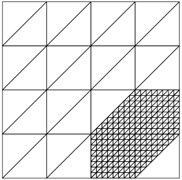
- applications : flow in porous media, materials with microstructures
- goal : efficient w.r.t. problem size, coefficients
- method
  - discretise $\longrightarrow$ large system of equations, ill-conditioned
  - domain decomposition : divide problem into many small subproblems
  - two-level method : additionally solve coarse problem

Robust Coarse Spaces



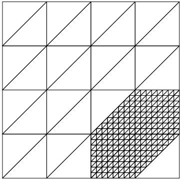
- spanned by set of basis functions
- coefficient explicit convergence theory
- basis functions have to sum up to one (partition of unity)
- basis functions should have low energy, i.e., good basis functions *flat* in high coefficient regions

Multiscale Finite Element Basis



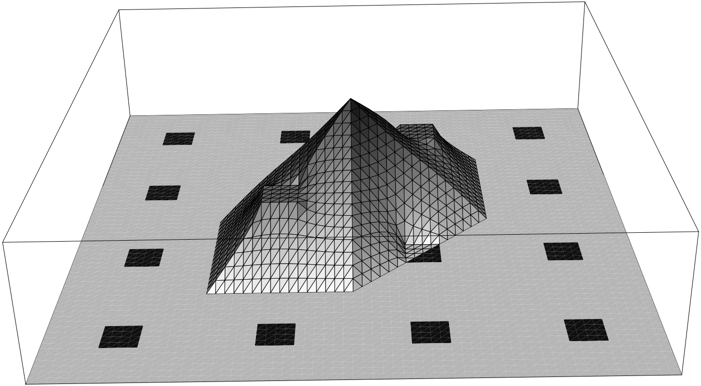
- local problems on *coarse triangles*
- partition of unity by
  "well chosen" boundary conditions
- fairly *ad-hoc*
- need *coarse grid*

Energy Minimizing Basis

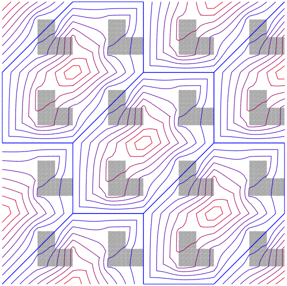


- local problems on *overlapping patches*
- *zero* boundary conditions
- partition of unity constraint
- equivalent to *constrained energy minimisation*
- global system of *same size* as original
- but very structured, can be done *efficiently*

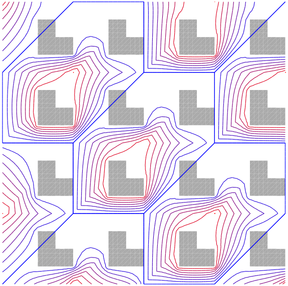
Energy Minimizing Basis Function Example



Coarse Basis Examples



multiscale finite element basis



energy minimizing basis

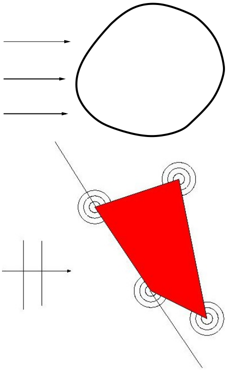
Robust Coarse Spaces

- pictures from recently submitted paper (BICS preprint 12/08)
- future:
  - analysis for construction
  - adaptive choice of patches

Domain Decomposition and Multiscale Methods: Links and Collaborations

- **Eero Vainikko (Tartu)** parallel implementation
- **Clemens Pechstein (Linz)** FETI methods, detailed analysis, 1 paper in press, 2 in preparation
- **Tom Hou (Caltech)** multiscale methods
- **Burak Aksoylu (Lousiana State), Hector Klie (Austin)** algebraic preconditioning for high-contrast diffusion, 1 paper
- **Ludmil Zikatanov (Penn State), Panayot Vassilevski (LLNL)** analysis for construction
- **Ian Sloan, Frances Kuo (UNSW)** stochastic PDEs, (Quasi) Monte Carlo methods

High Frequency Scattering



- applications: radar, acoustics, ultrasound therapy
- boundary element methods
- exploit asymptotic theory (Theme A)
- **Reading, BAE Systems, ICR, Met Office, Schlumberger**
- **Tatiana Kim (phd)**
- postdocs at Bath and Reading

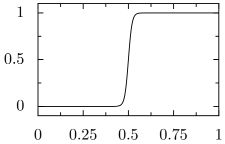
Other PhD Projects

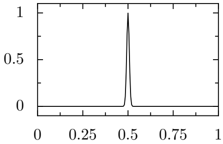
- **Stefano Gianni (phd)** recently finished
  adaptivity for eigenvalue solvers
- **Richard Norton (phd)** about to submit
  photonic crystal fibres
- **Fynn Scheben (phd)**
  neutron transport, Serco Assurance
- **Chris Hart (phd)** with Marco Marletta (Cardiff)
  localised modes for Schrödinger in exterior domain
- **Emily Dodgson (phd)** with Andrew Rees (Mech Eng)
  thermoconvective instabilities in porous media

Numerics for Weather and Climate Models

- **Mike Cullen (Met Office)**
- **Melina Freitag (GWR fellow)** data assimiliation
- **Sean Buckeridge (phd)** multigrid methods
- **Emily Walsh (phd)** adaptive methods
- **Ian Sloan (UNSW)** approximation on sphere

Adaptive Methods for PDEs





- problems with boundary layers, shocks, blowup, corner singularities
- more points where needed
- equidistribute *monitor function*
- h-adaptive methods: add/remove points
- p-adaptive methods: increase degree
- r-adaptive methods: move points
  couple equation for points to PDE

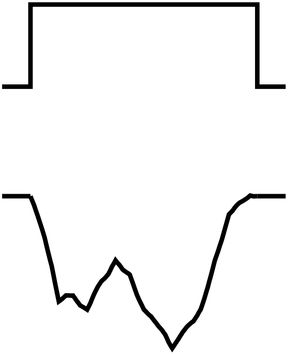
Monitor Function

$$M(\tilde{x}, \tilde{y}) \qquad M(u(\tilde{x}, \tilde{y}))$$

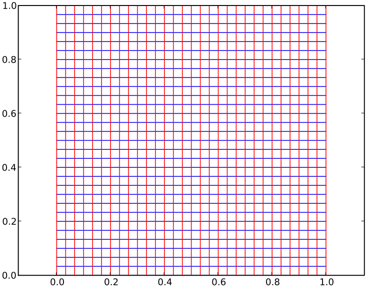
- indication of point density
- 1D
  - equal intervals $\longrightarrow$ intervals with given length
  - mapping unique
- 2D
  - equal squares $\longrightarrow$ quads with given area
  - mapping not unique
- idea: minimize displacement of points

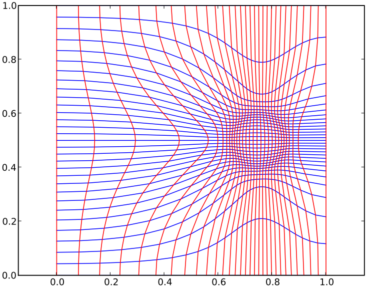
Optimal Mass Transport

- Gaspard Monge (1781)
- pile of ruble and hole
- efficient strategy for moving
- mapping $(x, y) \longrightarrow (\tilde{x}, \tilde{y})$
- monitor function: shape of hole
- minimise mean square displacement
- Leonid Kantorivich (1942)



2D Example





Monge-Ampére Equation

- under certain reasonable conditions
- map is gradient of (convex) potential P

$$(x, y) \rightarrow (\tilde{x}, \tilde{y}) = (P_x, P_y)$$

- change in area = determinant of Jacobian

$$\begin{vmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{vmatrix} = (P_{xx}P_{yy} - P_{xy}^2)$$

- Monge-Ampère equation

$$(P_{xx}P_{yy} - P_{xy}^2)M(P_x, P_y) = \theta$$

Solving Monge-Ampére Equations

- Parabolic Monge-Ampère: potential issues: parameters, stiffness, choice of artifical time, natural coupling
- Newton-Krylov-Multigrid: fairly standard approach, nonlinear, need to worry about convexity (related to mesh tangling) and global convergence,
- nonlinear multigrid: standard nonlinear multigrid scheme, specially tailored smoother, in d dimensions only need solve polynomials of order d, still need to worry about convexity
- linear programming

Plan

- preliminary 2D implementation (python)
  - parabolic relaxation
  - Gauss-Newton-Krylov
- 2D and 3D implementation in PETSc, with preconditioning
- couple with nonlinear problems (blowup, chemotaxis, phase field, Gierer-Meinhardt, Cahn-Hilliard, Allen-Cahn, Schrödinger)