Robust Coarsening for Domain Decomposition Methods

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Overview

- elliptic equation with variable coefficient $\alpha > 0$

$$\nabla \cdot (\alpha \nabla u) = f$$

- finite element discretization
- system of equations

$$Au = f$$

- preconditioned conjugate gradient
- one-level domain decomposition preconditioner
- two-level domain decomposition preconditioner
- how to construct the second level?

Solving the System of Equations

- system of equations

$$Au = f$$

- A is symmetric positive definite
- A is large, but sparse and structured
- 1D, linear elements: tridiagonal
- 2D, regular grid, linear elements:
  block tridiagonal with tridiagonal blocks
- direct solvers for 1D, maybe 2D, not 3D
- constant coefficients: (block-)Toeplitz, FFT
- unstructured grids, varying coefficients:
  multilevel iterative methods

Iterative Methods

- preconditioned Richardson method

$$u^{k+1} = u^k + B(f - Au^k)$$

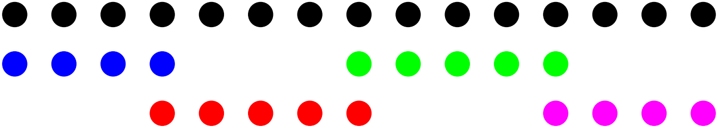
- convergence if $\rho(I - BA) < 1$
- preconditioned conjugate gradient method
- convergence determined by $\kappa(BA)$
- scalable and robust methods:
  number of iterations and cost per iteration well behaved w.r.t.
  - problem size, mesh resolution
  - number of subdomains
  - coefficients!
- ideally for N unknowns:
  $O(1)$ iterations, $O(N)$ operations per iteration

Domain Decomposition Methods

- whole system too much for direct solver (or 1 computer)
- decompose the problem into smaller subproblems
- subproblems are coupled : iteration
- divide domain into smaller subdomains
- many different types
- here overlapping additive Schwarz method

Restriction Matrices (1D)

- overlapping subdomains



- restriction matrices $R_i = \square$

$$
R_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

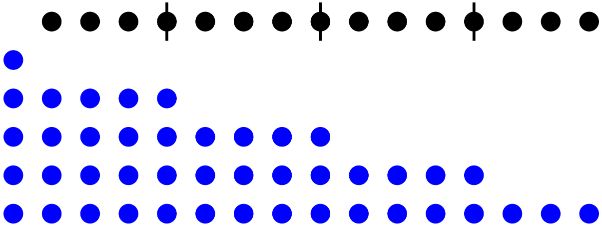
- extension matrices $R_i^T = \square$

Formulation of the One-Level Method

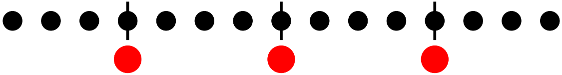
- restriction of whole space to subspace i : $R_i = \square$
- extension from subspace i into whole space : $R_i^T = \square$
- matrix for subproblem $R_i A R_i^T = \square \, \square \, \square = \square$
- for injection, A_i is submatrix of A
- preconditioner
  $y = Bx = \sum_i R_i^T A_i^{-1} R_i x$
  $\big| = (\square \, \square^{-1} \, \square + \square \, \square^{-1} \, \square + \cdots) \big|$

Convergence of the One-Level Method

- not scalable, illustrate with 1D problem
- rhs $f = 0$, BC $u(0) = 1$, $u(1) = 0$, start with $u^0 = 0$
- information moves 1 subdomain per iteration



- number of iterations depends on number of subdomains
- remedy: in addition to local solves, do 1 global solve



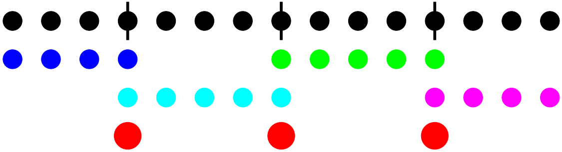
Formulation of the Two-Level Method

- fine level: subproblems that cover the whole problem

$$B = \sum_i R_i^T A_i^{-1} R_i$$

- coarse level: one smaller problem for whole domain

$$\hat{B} = R_0^T A_0^{-1} R_0$$



- choice of coarse problem
  - one unknown from each subdomain
  - average unknowns in one subdomain
  - weighted average, linear basis functions

Formulation of the Two-Level Method

- system $Au = f$
- restriction matrices R_i
- local problems $A_i = R_i A R_i^T$
- one-level preconditioner

$$B = \sum_i R_i^T A_i^{-1} R_i$$

- coarse problem $A_0 = R_0 A R_0^T$
- two-level preconditioner

$$\tilde{B} = R_0^T A_0^{-1} R_0 + \sum_i R_i^T A_i^{-1} R_i$$

- columns r_i of R_0^T represent coarse basis functions

Construction of the Coarse Space

- basis functions defined on subdomains $r_i = R_i^T q_i$
- solution of local problem $A_i q_i = g_i$
- well chosen right hand side g_i
- assume $g_i = R_i g \quad \Rightarrow \quad r_i = R_i^T A_i^{-1} R_i g$
- preservation of constants $\sum_i r_i = \mathbf{1}$

$$\sum_i r_i = \sum_i R_i^T A_i^{-1} R_i g = Bg = \mathbf{1}$$

- g corresponds to the Lagrange multipliers of a constrained minimization problem (Wan, Chan, Smith 2000) (Xu, Zikatanov 2004)
- how to solve the system $Bg = \mathbf{1}$?

Preconditioning the One-Level Preconditioner

- precondition B with A

$$\kappa(AB) = \kappa(BA)$$

- only as good as one-level method
- B has special structure, "local" operator
- no global solve needed
- construct one-level preconditioner for B (hinted at in Zikatanov, Xu 2004)
- other ideas
  - diagonal preconditioner : $D = \text{diag}(B)^{-1}$
  - localized version of A : $E = \sum_i R_i^T A_i R_i$

One-Level Preconditioner for the One-Level Preconditioner

- matrix A
- one-level preconditioner $B = \sum_i R_i^T A_i^{-1} R_i$
- local problems for $A_i = R_i A R_i^T$
- A_i is sparse
- one-level preconditioner $C = \sum_j R_j^T B_j^{-1} R_j$
- local problems $B_j = R_j B R_j^T$
- B_j is dense
- $B \sim A^{-1}$ and $C \sim B^{-1}$ so somehow $C \sim A$

Implementing the Preconditioner

- consider a domain j with 2 neighbors k and l

$$R_j R_j^T = I_j, \qquad R_j R_k^T = \hat{I}_{jk} \neq 0, \qquad R_j R_l^T = \hat{I}_{jl} \neq 0$$

- local problem j

$$\begin{aligned} B_j &= R_j B R_j^T \\ &= R_j \left(\sum_i R_i^T A_i^{-1} R_i\right) R_j^T \\ &= A_j^{-1} + \hat{I}_{jk} A_k^{-1} \hat{I}_{kj} + \hat{I}_{jl} A_l^{-1} \hat{I}_{lj} \end{aligned}$$

- all A_i^{-1} are dense
- how can we efficiently apply B_j^{-1}?

Linear Algebra Trick

- local problem solve

$$B_j^{-1} = (A_j^{-1} + \hat{I}_{jk} A_k^{-1} \hat{I}_{kj} + \hat{I}_{jl} A_l^{-1} \hat{I}_{lj})^{-1}$$

- apply Sherman-Morisson-Woodbury formula

$$(A^{-1} + U\Sigma^{-1}V^T)^{-1} = A - AU(\Sigma + V^T AU)^{-1} V^T A$$

- set $A \leftarrow A_j, \quad U = V \leftarrow \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix}, \quad \Sigma \leftarrow \begin{bmatrix} A_k & \\ & A_l \end{bmatrix}$

$$B_j^{-1} = A_j - A_j \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix} \left(\begin{bmatrix} A_k & \\ & A_l \end{bmatrix} + \begin{bmatrix} \hat{I}_{kj} \\ \hat{I}_{lj} \end{bmatrix} A_j \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix} \right)^{-1} \begin{bmatrix} \hat{I}_{kj} \\ \hat{I}_{lj} \end{bmatrix} A_j$$

- sparse system solve

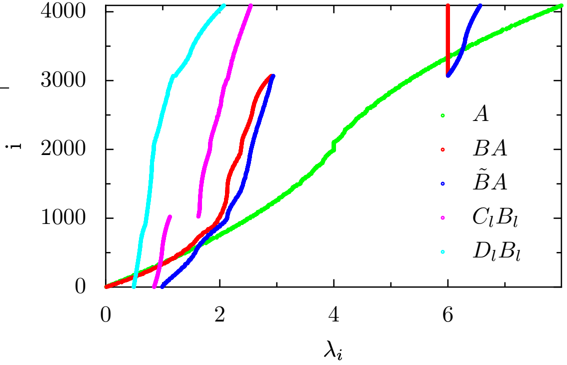
Efficiency and Robustness

- number of iterations : $\kappa(CB)$
- cost of C : multiple of cost of B
- constants depend only on
  number of neighbors of subdomains,
  not on number of domains or coefficients
- therefore constructing R_0 is scalable and robust

Spectral Analysis (a)

constant coefficients, small domains ($\sim$ multigrid)
$n = (2,2)$, $d = 1$, $\min\alpha = 1.0$, $\max\alpha = 1.0$,
$n_p = (32,32)$

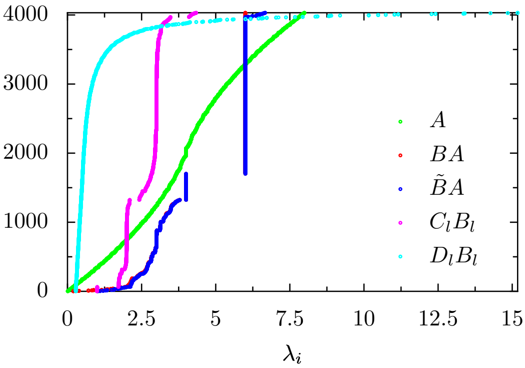
	$\lambda_{\max}$	$\lambda_{\min}$	κ
A	$7e0$	$9e-3$	$8e2$
BA	$6e0$	$1e-2$	$5e2$
$\tilde{B}A$	$6e0$	$9e-1$	$6e0$
C_lB_l	$2e0$	$8e-1$	$2e0$
D_lB_l	$2e0$	$4e-1$	$4e0$



Spectral Analysis (b)

constant coefficients, large domains
$n = (8, 8)$, $d = 4$, $\min \alpha = 1.0$, $\max \alpha = 1.0$, $n_p = (8, 8)$

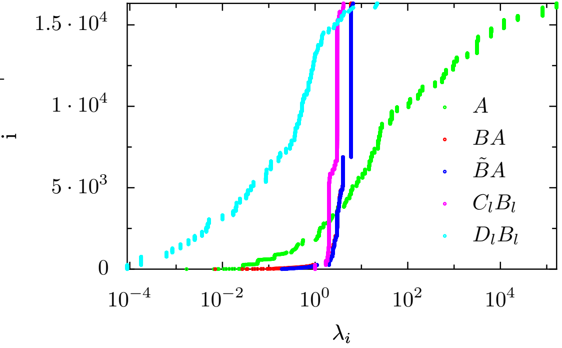
	$\lambda_{\max}$	$\lambda_{\min}$	κ
A	$7e0$	$9e-3$	$8e2$
BA	$6e0$	$1e-1$	$3e1$
$\tilde{B}A$	$6e0$	$1e0$	$6e0$
C_lB_l	$4e0$	$9e-1$	$4e0$
D_lB_l	$1e1$	$2e-1$	$5e1$



Spectral Analysis (c)

varying coefficients
$n = (8, 8)$, $d = 4$, $\sigma = 4.0$, $\min \alpha = 5e-6$,
$\max \alpha = 3e3$, $n_p = (8, 8)$

	$\lambda_{\max}$	$\lambda_{\min}$	κ
A	$4e5$	$2e-2$	$2e7$
BA	$6e0$	$8e-2$	$7e1$
$\tilde{B}A$	$6e0$	$8e-1$	$7e0$
$C_l B_l$	$6e0$	$9e-1$	$6e0$
$D_l B_l$	$2e1$	$8e-5$	$3e5$



Summary

- considered elliptic equations with varying coefficients
- two-level preconditioner
  for a given set of overlapping subdomains
- construction is not cheap, but algebraic, scalable and robust
- main ideas
  - one-level preconditioner for one-level preconditioner
  - linear algebra trick
- topics for further research
  - analysis of $\kappa(CB)$
  - for overall scalability and robustness,
    it is important to choose the subdomains well
  - non-symmetric systems

References

- Mandel, Brezina, Vaněk, *Energy Optimization of Algebraic Multigrid Bases* (1999)
- Wan, Chan, Smith, *An Energy-Minimizing Interpolation for Robust Multigrid Methods* (2000)
- Xu, Zikatanov, *On an Energy Minimizing Basis for Algebraic Multigrid Methods* (2004)
- Graham, Lechner, Scheichl, *Domain Decomposition for Multiscale PDEs* (2006)
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Convergence of the Two-Level Method

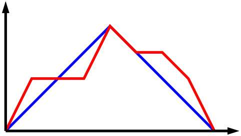
- coefficient explicit theory for overlapping Schwarz (Scheichl & Vainikko 2006)
- coarse space robustness indicator

$$\gamma(\alpha) = \max_i \delta_i^2 \|\alpha |\nabla \Phi_i|^2\|_{L_\infty}$$

- condition number bound

$$\kappa(\tilde{B}A) \lesssim \gamma(\alpha) \left(1 + \max_i \frac{H_i}{\delta_i}\right)$$

- we want Φ_i that are flat where α is high



Energy Minimizing Coarse Space Basis

- from the theory we know that
  - energy of basis functions must be low $\|\alpha|\nabla\Phi_i|^2\|_{L_\infty}$

$$r_i^T A r_i = \|r_i\|_A^2$$

  - basis functions must preserve constants $\sum_i \Phi_j = 1$

$$\sum_i r_i = \mathbf{1}$$

- constrained minimization problem

Constrained Minimization Problem

- (Wan, Chan, Smith 2000)

$$
\begin{aligned}
\min \ & \sum_i r_i^T A r_i = \operatorname{tr} R_0 A R_0^T = \operatorname{tr} A_0 \\
\text{s.t.} \ & \sum_i r_i = R_0 \mathbf{1} = \mathbf{1} \\
& r_i = R_i^T q_i
\end{aligned}
$$

- solve using Lagrange multipliers