Space-Time Multigrid for Anisotropic Partial Differential Equations

Jan Van lent

K.U.Leuven

Department of Computer Science

Celestijnenlaan 200A

B-3001 Leuven

Belgium

Overview

☞ Overview

- Model Problems
- Discretisation
- Smoothing
- Multigrid
- Fourier-Laplace Analysis
- Results
- Conclusion

Time-Dependent Problems

- parabolic PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f$$

- elliptic PDE

$$\mathcal{L}u + f = 0$$

Model Problems

- Diffusion Equation

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Anisotropic Diffusion Equation

$$\mathcal{L} = \varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Rotated Anisotropic Diffusion Equation

$$\begin{aligned}\mathcal{L} =& (\varepsilon c^2 + s^2) \frac{\partial^2}{\partial x^2} + (c^2 + \varepsilon s^2) \frac{\partial^2}{\partial y^2} + \\ & 2(\varepsilon - 1) cs \frac{\partial^2}{\partial x \partial x}\end{aligned}$$

$$c = \cos\beta,\, s = \sin\beta$$

Discretisation

- method of lines (spatial discretisation)
  - approximate by grid of continuous functions
  - finite difference approximation of operator
  - system of ODEs

$$\dot{u} = Lu + f$$

  - e.g., anisotropic diffusion equation
    (stencil notation)

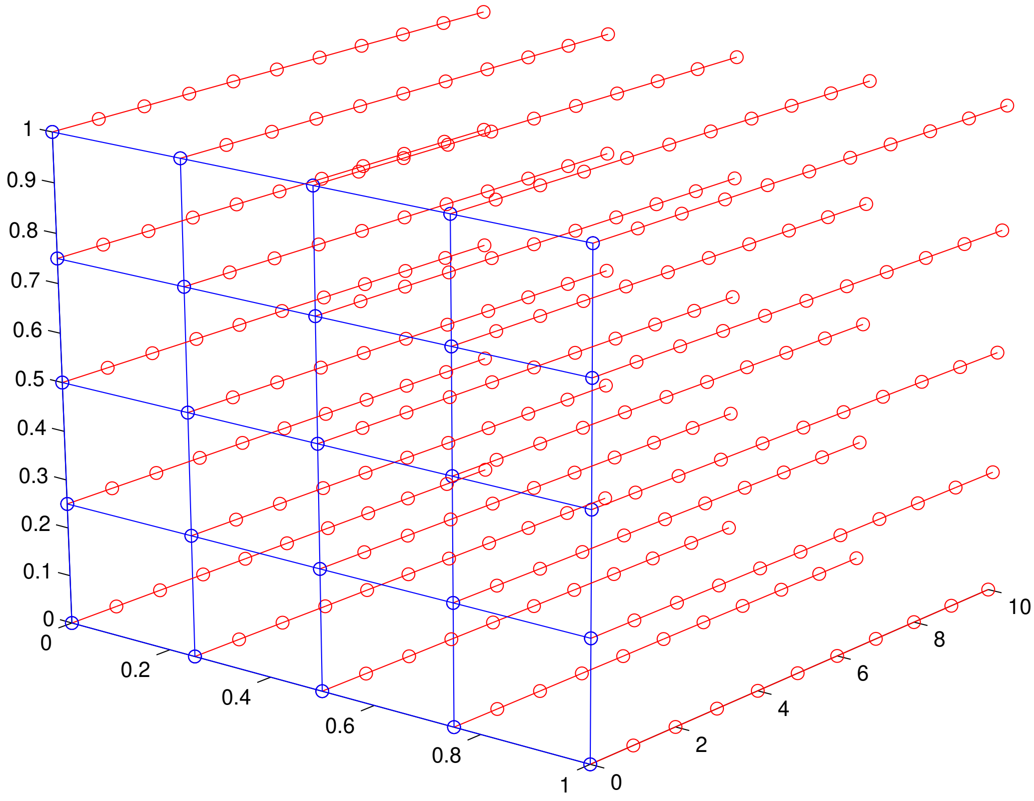
$$L = \begin{bmatrix} & \frac{1}{h_y^2} & \\ \frac{\varepsilon}{h_x^2} & -\frac{2\varepsilon}{h_x^2} - \frac{2}{h_y^2} & \frac{\varepsilon}{h_x^2} \\ & \frac{1}{h_y^2} & \end{bmatrix}$$

    rotated equation : 9-point stencil

- approximate solution of ODEs $\rightarrow$ time discretisation
  - approximate by grid of discrete functions
  - stiff problems $\rightarrow$ implicit solvers

Grid

- grid of continuous or discrete functions



Smoothing

- extend standard iterative methods to system of ODEs

$$\dot{u}^{(\nu+1)} = Pu^{(\nu+1)} + Qu^{(\nu)} + f$$

  where $L = P + Q$ is a standard splitting

- scalar equation per point $\rightarrow$ ODE per point

  point relaxation : Jacobi, Gauss-Seidel

- line relaxation : horizontal and vertical line Jacobi, Gauss-Seidel, zebra Gauss-Seidel

Can be interpreted as

- line relaxation in time dimension
- (x, t) or (y, t) plane relaxation

Examples

- Jacobi

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu)} + f$$

- Lexicographic Gauss-Seidel

$$\dot{u}^{(\nu+1)} = \left[\ \bullet\ \right] u^{(\nu+1)} + \begin{bmatrix} & & \\ \bullet & & \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ & & \bullet \\ & & \end{bmatrix} u^{(\nu)} + f$$

- Red-Black Gauss-Seidel

$i + j$ even

$$\dot{u}^{(\nu+1)} = \left[\ \bullet\ \right] u^{(\nu+1)} + \begin{bmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \end{bmatrix} u^{(\nu)} + f$$

$i + j$ odd

$$\dot{u}^{(\nu+1)} = \left[\ \bullet\ \right] u^{(\nu+1)} + \begin{bmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \end{bmatrix} u^{(\nu+1)} + f$$

- Four Colour Gauss-Seidel

- Horizontal Zebra Gauss-Seidel

i even

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ & & \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu)} + f$$

i odd

$$\dot{u}^{(\nu+1)} = \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + \begin{bmatrix} \bullet & \bullet & \bullet \\ & & \\ \bullet & \bullet & \bullet \end{bmatrix} u^{(\nu+1)} + f$$

- Vertical Zebra Gauss-Seidel

- Alternating Zebra Gauss-Seidel

Multigrid

- system $\dot{u}_h = L_h u_h + f_h$
- algorithm

pre smoothing solve

$$\dot{v}_h^{(1)} = P_h v_h^{(1)} + Q_h v_h^{(0)} + f_h$$

defect calculation $d_h = L_h v_h^{(1)} + f_h - \dot{v}_h^{(1)}$

residual equation solve recursively

$$\dot{e}_H = L_h e_H + I_h^H d_h$$

correction $v_h^{(2)} = v_h^{(1)} + I_H^h e_H$

post smoothing solve

$$\dot{v}_h^{(3)} = P_h v_h^{(3)} + Q_h v_h^{(2)} + f_h$$

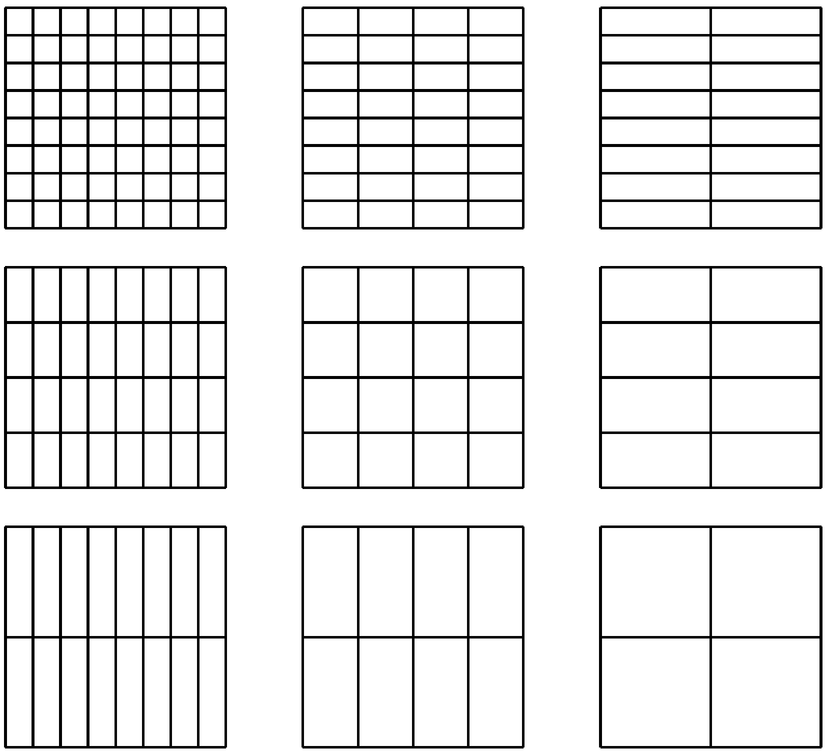
[Lubich and Ostermann, 1987; Vandewalle, 1993]

Smoothing

- point relaxation
- line relaxation for anisotropic problems

Coarsening

- standard coarsening : double grid spacing in both directions
- semicoarsening for anisotropic problems
  - x-semicoarsening, y-semicoarsening
  - alternating semicoarsening
    (MG-S [Washio and Oosterlee, 1995, 1998])



Fourier-Laplace Analysis

- error of successive approximations

$$e^{(\nu+1)} = \mathcal{M} e^{(\nu)}$$

- asymptotic convergence factor of $\mathcal{M}$ and its Laplace transform $M(z)$ are related as follows

$$\rho(\mathcal{M}) = \max_{z \in \Sigma} \rho(M(z))$$

- $M(z)$ is iteration operator of Laplace transform of the original equation

$$(zI - L)u = f$$

- spectral picture
- Σ depends on treatment of time dimension
  - continuous or discrete
  - finite time, infinite time or time periodic

[Vandewalle; Janssen; Ta'asan and Zhang]

Local Mode Fourier Analysis

- Two Grid Analysis
  - Standard Coarsening

$$M_s = S^{\nu_1} K_s S^{\nu_2}$$

$$K_s = I - I_H^h L_H^{-1} I_h^H L_h$$

  - Semicoarsening

$$M_x = S^{\mu_1} K_x S^{\mu_2} \qquad M_y = S^{\mu_1} K_y S^{\mu_1}$$

  - Alternating Semicoarsening

$$M_a = (M_x M_y)^{\nu_1} K_s (M_x M_y)^{\nu_2}$$

- operator transforms exponential Fourier modes

$$\rho(M(z)) = \max_{\theta} \rho(M(z,\theta))$$

[Vandewalle; Horton; Ta'asan and Zhang]

[Wesseling; Oosterlee]

- ✓ Overview
- ✓ Model Problems
- ✓ Discretisation
- ✓ Smoothing
- ✓ Multigrid
- ✓ Fourier-Laplace Analysis
- ☞ Results
- Conclusion

Results

Model Problem

- Rotated Anisotropic Diffusion Equation (ε, β)

Special Cases

- Anisotropic Diffusion Equation (ε, $\beta = 0$)
- Diffusion Equation ($\varepsilon = 1, \beta = 0$)

Smoothers

- Red-Black or Four-Colour Gauss-Seidel
- Horizontal, Vertical and Alternating Zebra Gauss-Seidel

Coarsening

- Standard Coarsening
- X- and Y-Semicoarsening
- Alternating Semicoarsening

Correction for computational work

- convergence factor $\rho \quad (R = \log(\rho))$
- method requires α "work units"
- n iterations $\rightarrow \rho^n$, $n\alpha$
- convergence factor "per work unit"

$$\tilde{\rho} = \sqrt[\alpha]{\rho} \qquad \tilde{R} = \frac{R}{\alpha}$$

- correction factor α

Point Relaxation	1
Line Relaxation	2
Standard Coarsening (V)	$\frac{4}{3}$
Semicoarsening (V)	2
Alternating Semicoarsening (V)	$\frac{16}{3}$

(V) : V-cycle

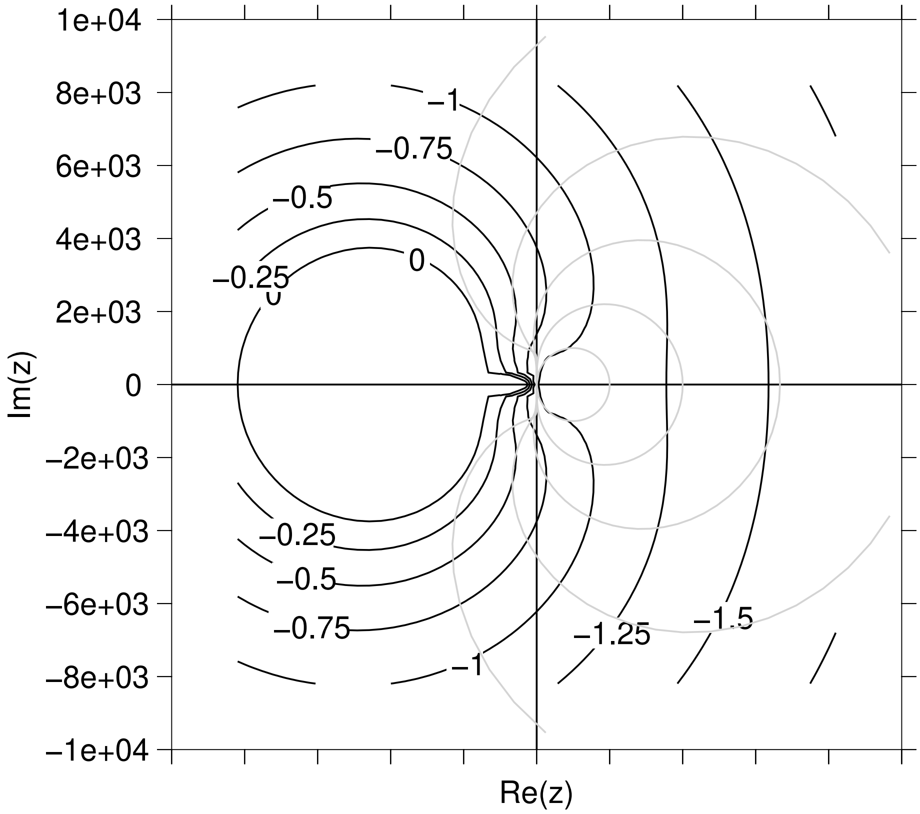
- e.g., Four Colour smoother with Alternating Semicoarsening (V)

$$\alpha = 2 \times 1 \times \frac{16}{3} = 10\frac{1}{3}$$

Spectral Image

- diffusion equation ($\varepsilon = 1, \beta = 0$)
- red-black Gauss-Seidel
- grid spacing $h = 2^{-5}$
- time step $h_t = 10^{-3}$
- boundary loci for BDF1-5

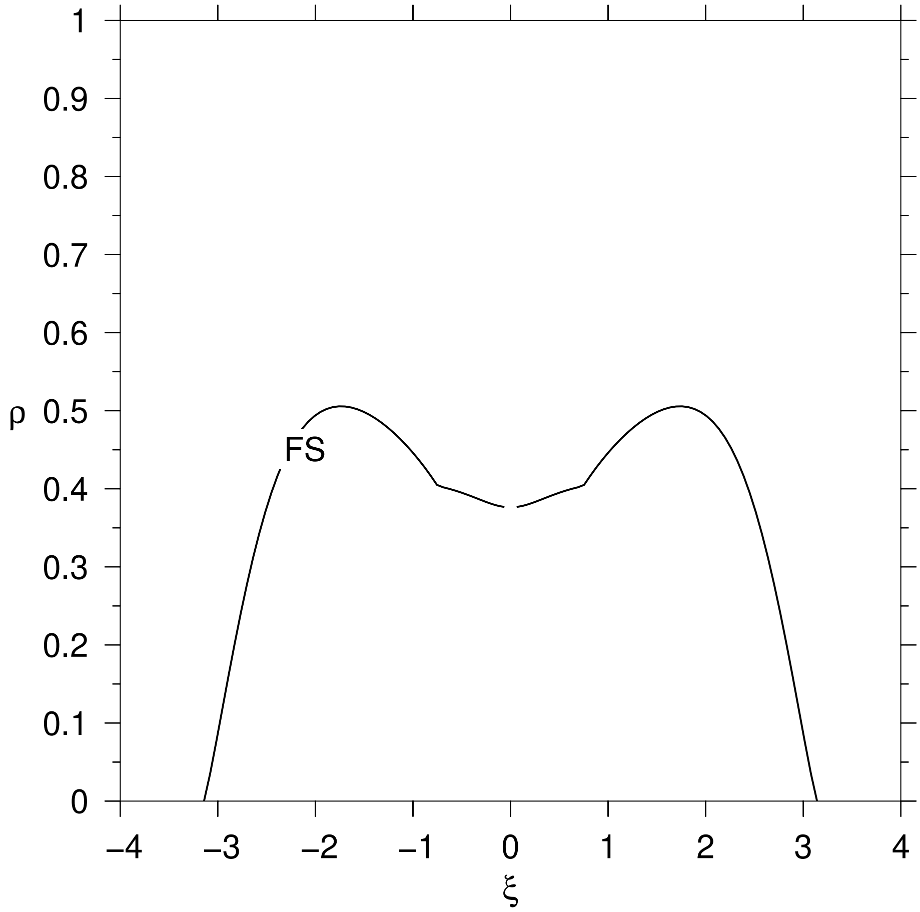
$$\rho(M(z))$$



$$h = 2^{-5}, \quad h_t = 10^{-3}, \quad \varepsilon = 1, \quad \beta = 0$$

$\rho(M(z))$ along the imaginary axis

$$z = \frac{1}{2}\frac{1-w}{1+w}, \quad w = e^{i\xi}$$



F : red-black or four-colour

H : horizontal zebra

V : vertical zebra

A : alternating zebra

S : standard coarsening

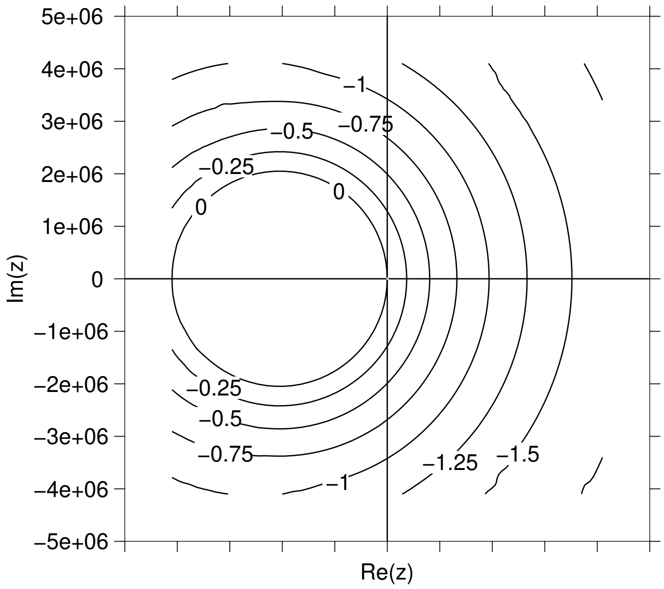
X : x-coarsening

Y : y-coarsening

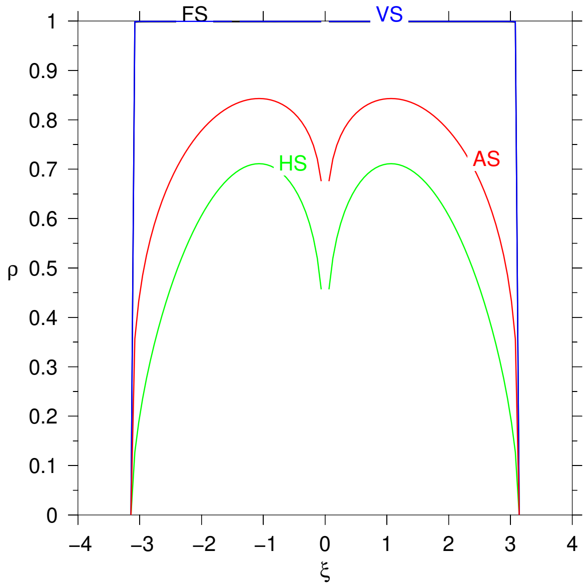
A : alternating semicoarsening

$$h = 2^{-5}, \quad h_t = 10^{-3}, \quad \varepsilon = 10^3, \quad \beta = 0$$

$\rho(M(z))$



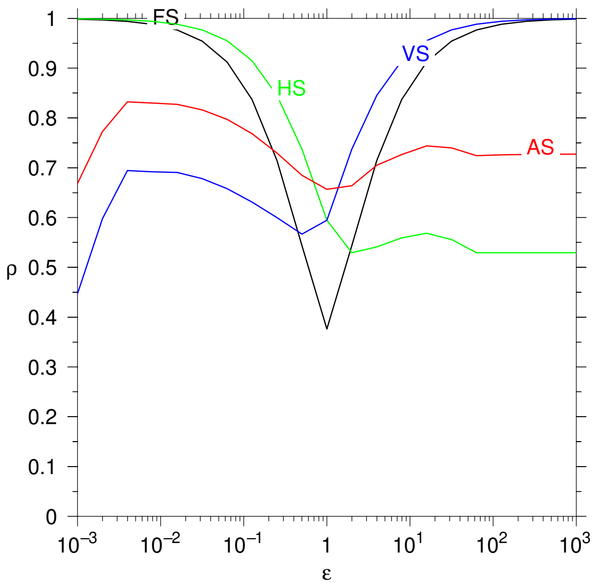
$\rho(M(z))$ along the imaginary axis



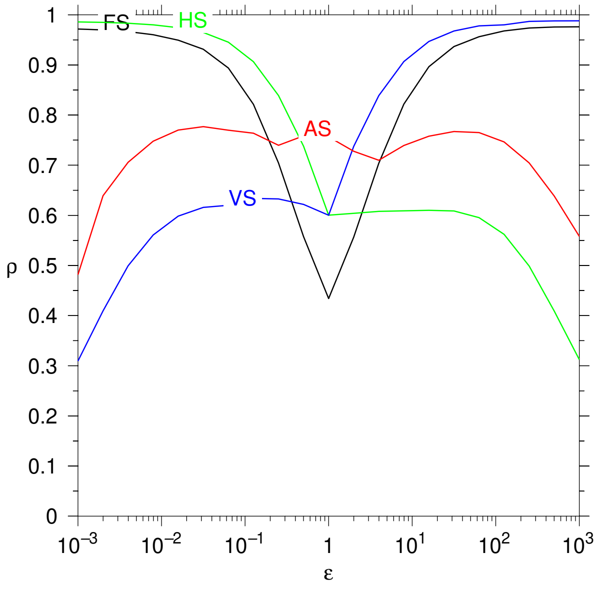
$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \beta = 0$$

$\rho(\varepsilon)$ for FS, HS, VS, AS

analysis



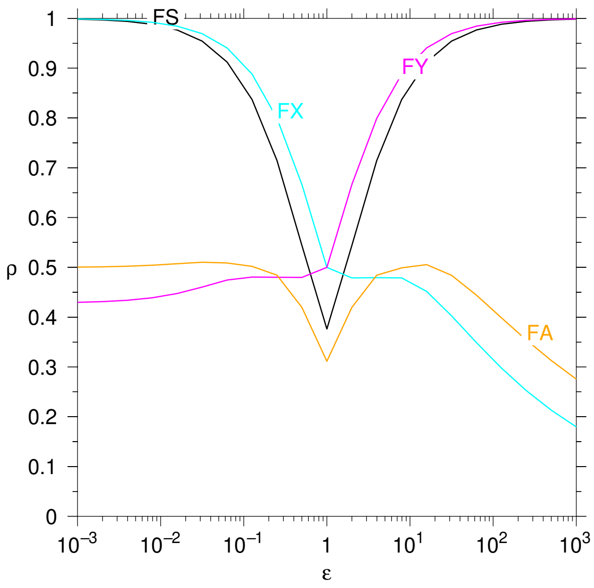
numerical



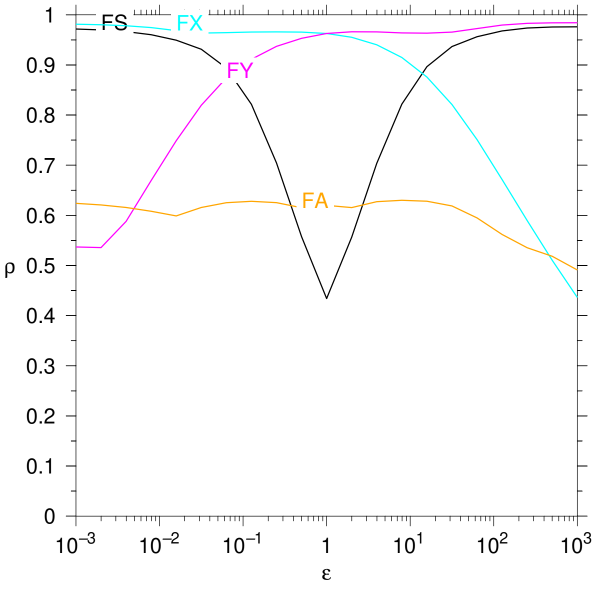
$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \beta = 0$$

$\rho(\varepsilon)$ for FS, FX, FY, FA

analysis



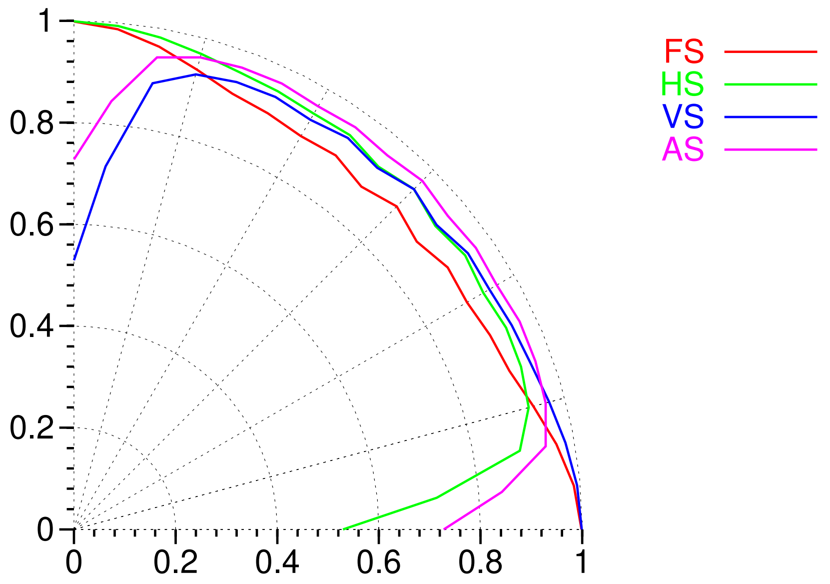
numerical



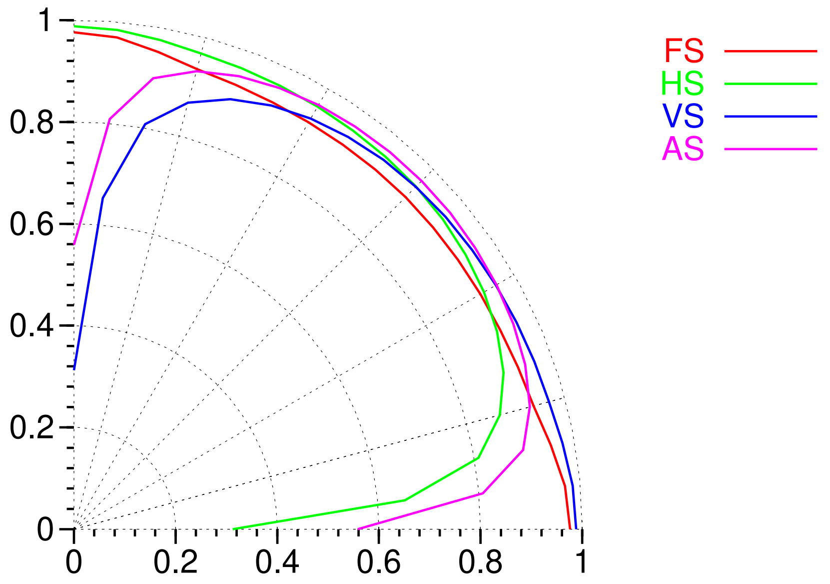
$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \varepsilon = 10^3$$

$\rho(\beta)$ for FS, HS, VS, AS

analysis



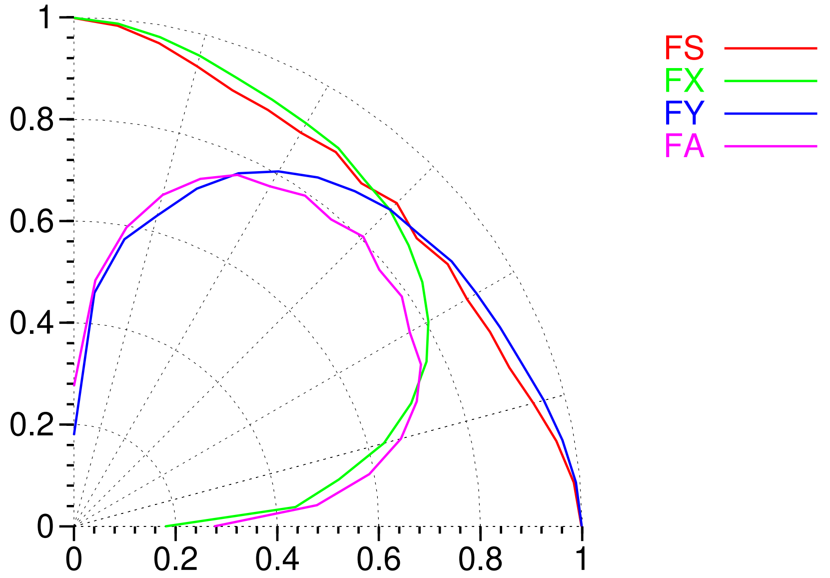
numerical



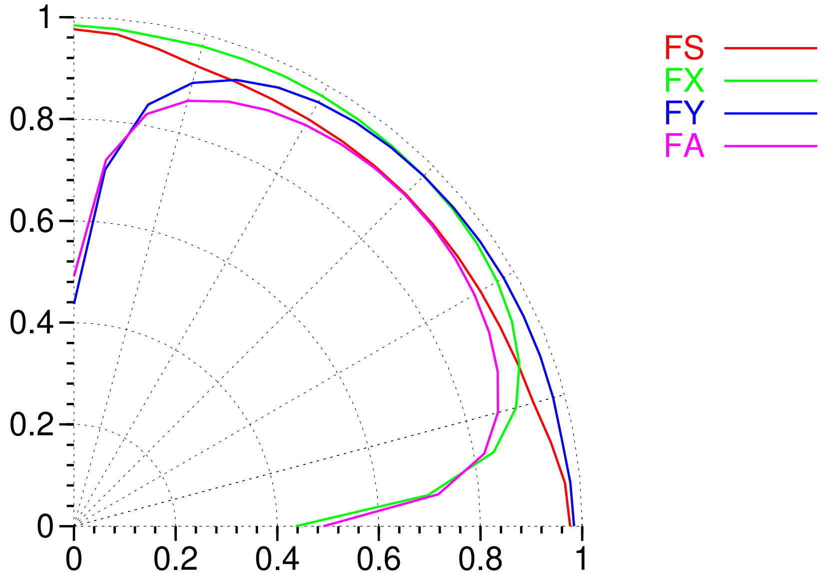
$$h = 2^{-5}, \quad h_t = 10^{-2}, \quad \varepsilon = 10^3$$

$\rho(\beta)$ for FS, FX, FY, FA

analysis



numerical



Variable Coefficients

- general diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(b\frac{\partial u}{\partial y}) + f$$

$a(x,y) = e^{10(x-y)}$, $b(x,y) = e^{-10(x-y)}$

- strong coupling in x *and* y directions
- convergence factor, (estimated) number of iterations

FS	HS	VS
$0.9207(>177)$	$0.9209(>178)$	$0.8926(>121)$

FX	FY
$0.8876(>121)$	$0.8867(>115)$

AS	FA	HY	VX
$0.0175(3)$	$0.0073(4)$	$0.1009(8)$	$0.1166(8)$

- ✓ Overview
- ✓ Model Problems
- ✓ Discretisation
- ✓ Smoothing
- ✓ Multigrid
- ✓ Fourier-Laplace Analyse
- ✓ Results
- ☞ Conclusion

Conclusion

- Standard relaxation $\rightarrow$ slow convergence
- Space-Time Multigrid
  - $\varepsilon = 1, \beta = 0$: red-black
  - $\varepsilon \neq 1, \beta = 0$: line smoothers, semicoarsening
  - $\varepsilon \neq 1, \beta \neq 0$: alternating zebra or semicoarsening
- methods for elliptic problems can be extended to parabolic problems

Future

- higher order time discretisation
- other test problems
- delay partial differential equations
- time-periodic solutions