Waveform Relaxation using Spectral Collocation in Time

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1. Waveform Relaxation
   - Classical Iterative Methods
   - Continuous and Discrete Waveform Relaxation
   - Convergence Analysis

2. Time Discretisation
   - Chebyshev Spectral Collocation
   - Stability and Convergence
   - Other Methods

3. Results
   - Model Problem
   - Convergence Analysis Results
   - Numerical Results

Classical Iterative Methods

- system $Ax = b$
- splitting $A = A^+ + A^-$
- iteration $A^+ x^{(\nu)} + A^- x^{(\nu-1)} = b$
- A^+ such that simple to solve
  - Jacobi: diagonal of A
  - Gauss-Seidel: lower triangular part of A
- iteration matrix $K = -(A^+)^{-1} A^-$
- spectral radius

$$\rho(K) = \max\{|\lambda| : \lambda \in \sigma(K)\}$$

- $\rho < 1 \Rightarrow$ convergence, the smaller the better

Continuous Waveform Relaxation

- iterative method for system of ODEs

$$\dot{x} = Ax + b$$

- same splitting $A = A^+ + A^-$

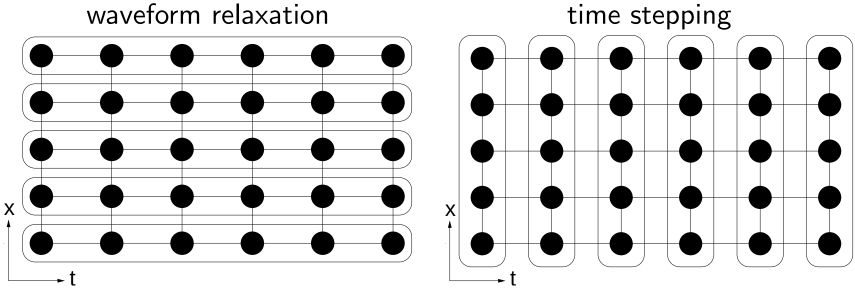
$$\dot{x}^{(\nu)} = A^+ x^{(\nu)} + A^- x^{(\nu-1)} + b$$

- e.g. Jacobi

$$\dot{x}_i^{(\nu)} + a_{ii} x_i^{(\nu)} = b_i - \sum_{j \neq i} a_{ij} x_j^{(\nu-1)}$$

Discrete Waveform Relaxation

- in practice: ODE integrator for each scalar ODE



- LMF, IRK, BVM have all been considered
- in this presentation Chebyshev Spectral Collocation

Convergence Analysis of Waveform Relaxation

- system of equations

$$(T \otimes I)x = (I \otimes A)x + b$$

- T : time derivative operator $w = Tv$
  - continuous $\quad w(t) = \frac{dv(t)}{dt}$
  - BDF1 $\quad w_i = \frac{v_i - v_{i-1}}{\Delta t}$
- $A = A^+ + A^- \longrightarrow$ iteration

$$(T \otimes I)x^{(\nu)} = (I \otimes A^+)x^{(\nu)} + (I \otimes A^-)x^{(\nu-1)} + b$$

- iteration operator

$$\mathcal{K} = (T \otimes I - I \otimes A^+)^{-1}(I \otimes A^-)$$

- spectral radius

$$\rho(\mathcal{K}) = \max_{z \in \sigma(T)} \rho(K(z))$$

- $K(z)$ iteration matrix for classical iterative method

$$K(z) = (zI - A^+)^{-1} A^-$$

- $\mathcal{K}$ non-normal operator $\rightarrow$ infinite time domains (alternatives: pseudospectrum, weighted norms)
- T operates on $v(t)$, $t \in [0, \infty)$ or v_i, $i \in 1..\infty$
- continuous WR on $[0, \infty) \rightarrow \sigma(T) = \overline{\mathbb{C}}^+$
- upper bound for A-stable methods $(\sigma(T) \subset \overline{\mathbb{C}}^+)$

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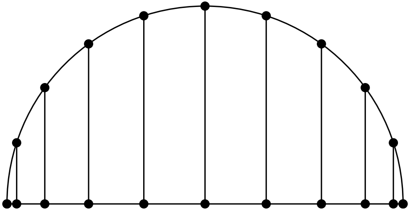
Chebyshev Spectral Differentiation

- interpolating spectral method (pseudospectral method)
- idea
  - given function $v(x)$
  - choose points x_j
  - construct interpolating polynomial $p(x_j) = v(x_j)$
  - approximate derivative $v'(x_j)$ by $w_j = p'(x_j)$

Chebyshev points

- interpolation $\rightarrow$ choice of x_j very important
- intuition: one-sidedness
  - central difference more accurate than forward or backward difference
  - points near boundary: more one-sided $\rightarrow$ higher density of points needed
- Chebyshev extreme points (Gauss-Chebyshev-Lobatto points)

$$x_j = \cos \frac{j\pi}{n}$$



Differentiation Matrix

- spectral differentiation: linear operation
  $\rightarrow$ can be written as matrix

$$\bar{\mathbf{w}} = \bar{\mathbf{D}}\bar{\mathbf{v}}$$

- simple explicit formula
- $[-1, 1] \rightarrow [0, \Delta t]$

$$t_j = \frac{1 - x_j}{2} \Delta t$$

$$\tilde{\mathbf{D}} = -\frac{2}{\Delta t} \bar{\mathbf{D}}$$

Chebyshev Spectral Collocation for ODEs

- replace $v'(t) = f(v(t))$ by $\tilde{\mathbf{D}}\tilde{\mathbf{v}} = f(\tilde{\mathbf{v}})$
- v_j parameters of unknown function
- partition $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{v}}$

$$\tilde{\mathbf{D}}\tilde{\mathbf{v}} = \left[\begin{array}{c|c} \cdot & \rule{1em}{2pt} \\ \hline \mathbf{d_0} & \mathbf{D} \end{array}\right] \left[\begin{array}{c} v_0 \\ \hline \mathbf{v} \end{array}\right]$$

- initial value condition $v(0) = v_0$
  $\rightarrow$ ignore first row of $\tilde{\mathbf{D}}$
- system of equation to solve

$$\mathbf{D}\mathbf{v} = -\mathbf{d_0} v_0 + f(\mathbf{v})$$

- many intervals $\rightarrow$ block CSC

$$\mathbf{D}\mathbf{v_i} = -\mathbf{d_0}\mathbf{e_n^T}\mathbf{v_{i-1}} + f(\mathbf{v_i})$$

Stability of block CSC

- Dahlquist test equation

$$v'(t) = \lambda v(t)$$

- solution $v(t) \to 0$ if $\lambda \in \mathbb{C}^-$
- discrete solution $v_i \to 0$ if $\lambda \in \frac{1}{\Delta t} S$
- stability domain

$$S = \{z \in \mathbb{C} : |R(z)| < 1\}$$

- stability function $(\Delta t = 1)$

$$R(z) = \mathbf{e_n^T}(z\mathbf{I} - \mathbf{D})^{-\mathbf{1}}\mathbf{d_0}$$

- a time discretization scheme is A-stable if $\mathbb{C}^- \subset S$

Stability and Convergence

- convergence of WR

$$\rho(\mathcal{K}) = \max_{z \in \sigma(T)} \rho(K(z))$$

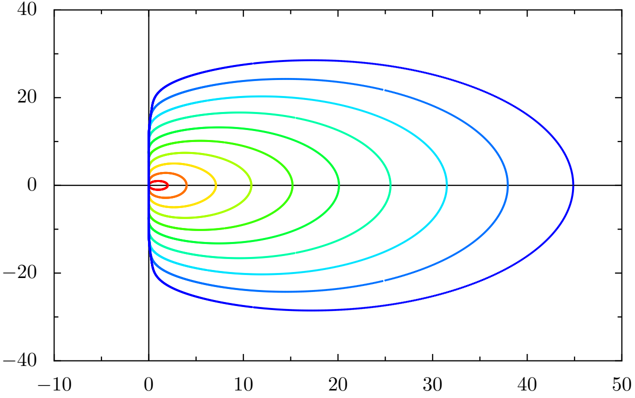
- continuous WR: $T = \frac{\partial}{\partial t}$, $\sigma(T) = \overline{\mathbb{C}}^+$
- discrete WR convergence is related to stability: $\sigma(T) = \mathbb{C} \backslash \frac{1}{\Delta t} S$
- A-stable scheme: $\mathbb{C}^- \subset S$ or $\sigma(T) \subset \overline{\mathbb{C}}^+$ or $\rho_{\rm disc} \leq \rho_{\rm cont}$
- CSC

$$\sigma(T) = \{z \in \sigma(\mathbf{D} + w\mathbf{e_n^T d_0}),\ |w| \leq 1\}$$

- boundary: set $w = e^{i\theta}$

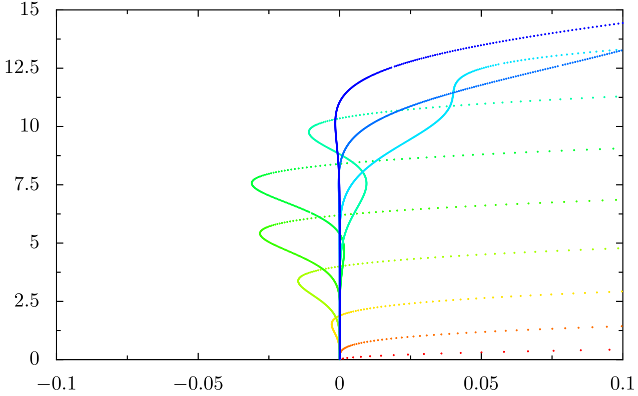
Stability Domains

CSC, $n = 1, \ldots 10$



A-stable?

A Closer Look

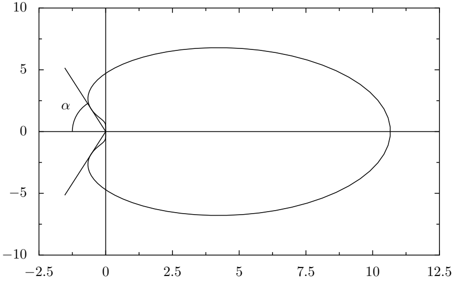


$A(\alpha)$-Stability

- $S_\alpha \subset S$

$$S_\alpha = \{|\arg(-z)| < \alpha\}$$

- BDF 3: $\alpha = 73^\circ$



α	BGBDF	BGAM	CSC
1	90.1756	90.0000	90.1756
2	90.0000	90.0000	90.0000
3	89.3188	90.0000	89.8903
4	87.7322	90.0000	89.7473
5	85.6488	90.0000	89.6995
6	83.0153	90.0000	89.7638
7	79.6949	90.0000	89.9361
8	75.9502	90.0000	89.9995
9	72.5365	86.7188	89.9971
10	69.5617	82.3122	89.9912

Other Methods

- many time discretization schemes exist
  - linear multistep methods: e.g. BDF
  - implicit Runge-Kutta methods:
    e.g. Gauss, Radau {I,II}A, Lobatto III{A,B,C}
  - (block) boundary value methods: e.g. GBDF, GAM
- many similarities
- polynomial collocation → IRK or BBVM
- Gauss, Radau and Lobatto methods: IRK based on Legendre points
- CSC: IRK based on Chebyshev points
- n-step GBDF on Gauss-Chebyshev-Lobatto points → CSC
- n-step GBDF on Gauss-Legendre-Radau points → Radau IIA

Features of CSC

- simple explicit formula for **D**
- if smooth → use high order → small number of unknowns
- spectral convergence
- all time steps simultaneously (like BVM)
- dense matrix
- expensive when applied mindlessly to large system of ODEs
- **but** in Jacobi and GS WR only scalar ODEs
- good stability for any order → good WR convergence
- for special cases fast transforms (FFT/DCT)
- potential problems with conditioning for high orders

Model Problem

- linear parabolic equation on unit square $(x, y) \in [0, 1]^2$

$$u_t = (au_x)_x + (bu_y)_y + cu + f$$

- finite difference in space (method of lines)
- stiff system of ODEs

$$\dot{u} = Lu + f$$

- L large sparse structured matrix
- Jacobi or Gauss-Seidel very inneficient
- adapt multigrid method for elliptic equation ($u_t = \dot{u} = 0$)

Example 1: Problem

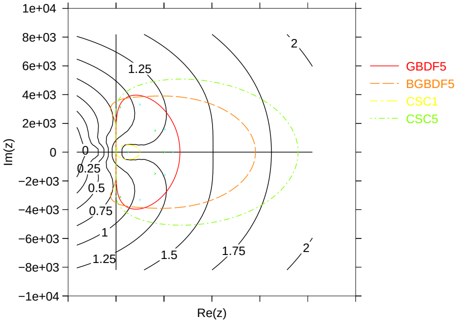
- diffusion equation ($a = b = 1$, $c = 0$)

$$u_t = u_{xx} + u_{yy} + f$$

- multigrid waveform relaxation
- convergence analysis
  - $\rho(\mathbf{M}(z))$ calculated using standard multigrid analysis
  - continuous WR on infinite intervals $z \in i\mathbb{R}$
  - CSC $z \in \sigma(\mathbf{D}) \subset \mathbb{C}^+$
  - block CSC on infinite sequences
    $z \in \sigma(\mathbf{D} + w\mathbf{d_0}\mathbf{e_n}^T), \quad |w| \leq 1$

Example 1: Convergence Analysis

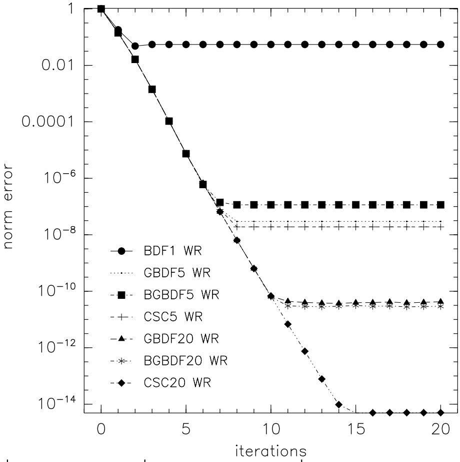
- $R(z) = -log_{10}\rho(\mathbf{M}(z))$
- $R = \min_{z\in\sigma(T)} R(z)$
  digits per iteration (asymptotic)
- spectral picture



GBDF5	BGBDF5	CSC1	CSC5
0.80	0.74	1.13	0.79

Example 1: Numerical Results

- discretisation error
  - $u(x, y, t) = sin(\frac{2\pi t}{10})$
  - $\Delta x = \Delta y = \frac{1}{8}$
  - time interval and #unknowns fixed
- convergence
  - $u(x, y, t) = 0$
  - $\Delta x = \Delta y = \frac{1}{32}$
  - $\rho^{(\nu)} = \frac{\|u^{(\nu)}\|}{\|u^{(\nu-1)}\|}$
  - $R^{(\nu)} = -\log_{10} \rho^{(\nu)}$



BDF1	GBDF5	BGBDF5	CSC5	GBDF20	BGBDF20	CSC20
0.97	0.87	0.87	0.87	0.11	0.10	0.88

Example 2

$$u_t = (au_x)_x + (bu_y)_y + cu + f$$

- $a(x, y, t) = e^{10(x-y)}$, $b(x, y, t) = e^{-10(x-y)}$, $c(x, y, t) = 0$, $u(x, y, t) = 0$
- anisotropic (coefficients depend on x and y)
- standard MG does not work
  (same for corresponding elliptic equation)
- adapt special MG methods to the parabolic case
- e.g. "multigrid as smoother", a multiple semicoarsening method
- convergence factor $\approx 6 \cdot 10^{-2}$

Concluding Remarks

- WR inspired by classical iterative methods
- WR convergence depends on stability of time discretisation
- if continuous WR converges on $[0, \infty)$
  $\rightarrow$ any A-stable ODE integrator will too
- for WR it is worth considering time discretisation schemes that would be too expensive in other cases
- CSC well suited for WR,
  if high order needed and possible (smooth problem)
- many time discretisation schemes are closely related

Further Reading

- WR convergence (Miekkala and Nevanlinna '87)
- IRK (Hairer and Wanner; Butcher; Wright '70)
- BVM (Brugnano and Trigiante; Iavernaro and Mazzia '98, '99)
- CSC (Trefethen; Boyd; Wright '64, '70)
- ODE/PDE/WR (Burrage)
- MG (Briggs, Henson and McCormick; Trottenberg, Oosterlee and Schüller)
- MG WR (Lubich and Ostermann '89; Vandewalle)
- MGS (Washio and Oosterlee '95, '98)