Constructing Robust Coarse Spaces for Overlapping Schwarz Methods

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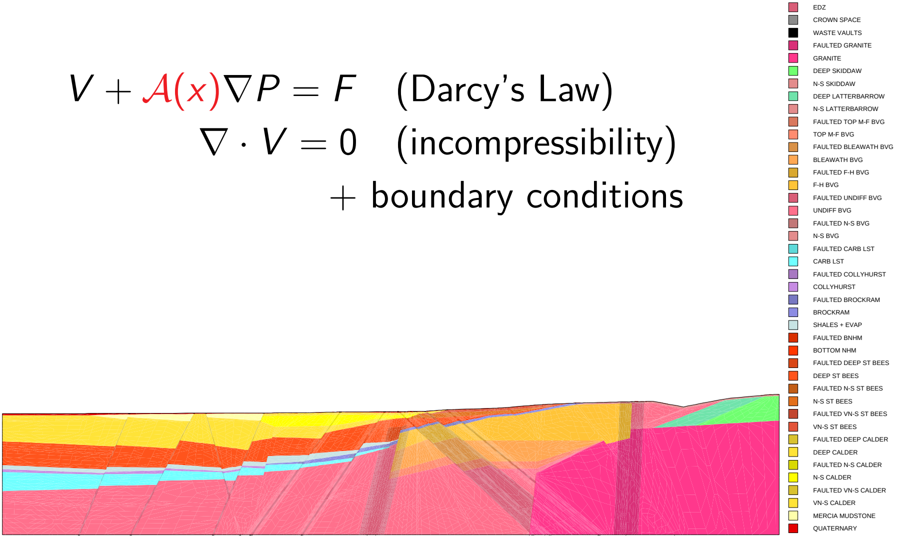
Motivation: Groundwater Flow (Sellafield)

$$V + \mathcal{A}(x)\nabla P = F \quad \text{(Darcy's Law)}$$

$$\nabla \cdot V = 0 \quad \text{(incompressibility)}$$

$$+ \text{ boundary conditions}$$



Model Problem

- elliptic PDE in 2D or 3D bounded domain Ω

$$-\nabla \cdot (\alpha \nabla u) = f \qquad (u = 0 \text{ on } \partial\Omega)$$

- highly variable (discontinuous) coefficients α
- finite element discretisation on mesh $\mathcal{T}^h$

$$Au = f$$

- very large and very ill-conditioned

$$\kappa(A) \lesssim \max_{\tau,\tau' \in \mathcal{T}^h} \left(\frac{\alpha_\tau}{\alpha_{\tau'}} \right) h^{-2}$$

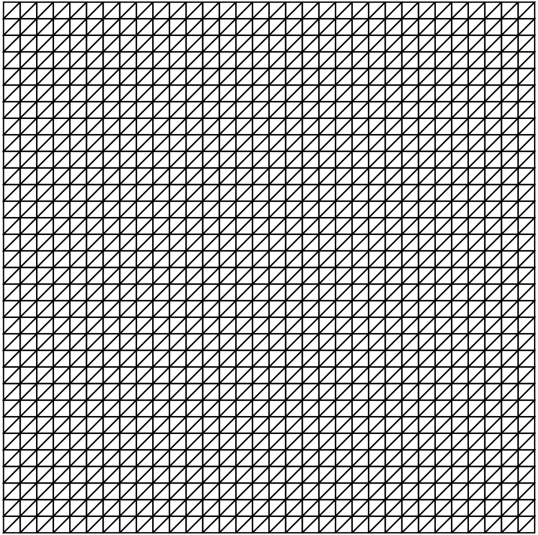
Goals

- unstructured grids, varying coefficients: multilevel iterative methods
- domain decomposition: two-level additive Schwarz method
- scalable and robust methods:
  - # iterations and
  - cost per iteration

  well behaved w.r.t.
  - problem size, mesh resolution
  - number of subdomains
  - coefficients!

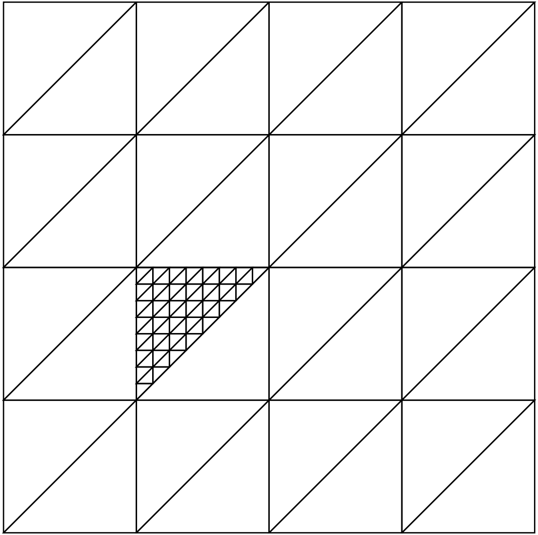
Domain Decomposition
Two-Level Overlapping Additive Schwarz Method

- fine grid



Domain Decomposition
Two-Level Overlapping Additive Schwarz Method

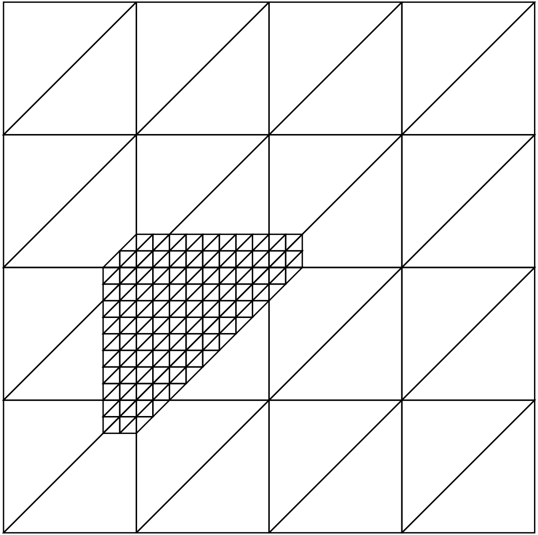
- fine grid
- subdomains



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

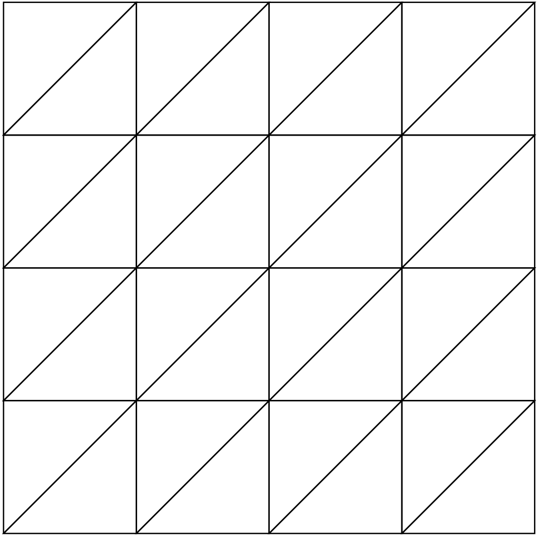
- fine grid
- subdomains
- overlap



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

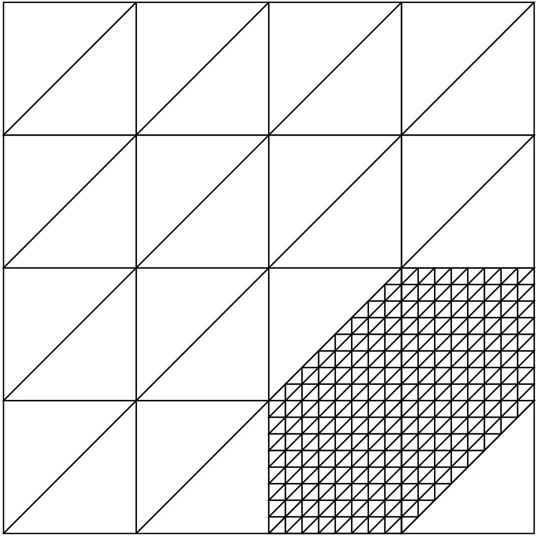
- fine grid
- subdomains
- overlap
- coarse grid



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

- fine grid
- subdomains
- overlap
- coarse grid
- coarse basis



Formulation of the Two-Level Method

- system matrix A
- restriction matrices R_i
- local problems $A_i = R_i A R_i^T$
- one-level preconditioner

$$B = \sum_i R_i^T A_i^{-1} R_i$$

- restriction matrix R_0
- coarse problem $A_0 = R_0 A R_0^T$
- two-level preconditioner

$$\tilde{B} = R_0^T A_0^{-1} R_0 + \sum_i R_i^T A_i^{-1} R_i$$

Choice of Coarse Space

- select coarse space basis $\{\Psi_j\}$ of finite element functions with $\sum_j \Psi_j \equiv 1$
- basis function Ψ_j represented by coefficient vector r_j
- vectors r_j columns of prolongation matrix R_0^T
- coarse space is spanned by columns of R_0^T
- basis function Ψ_j has support ω_j
- $\{\omega_j\}$ shape regular, uniformly overlapping, finite covering of Ω

Convergence Theory

Coefficient Explicit Condition Number Bound

- theorem [Scheichl, Vainikko, 2007]

$$\kappa(\tilde{B}A) \lesssim \gamma(\alpha) \left(1 + \max_j \frac{H_j}{\delta_j}\right)$$

  - H_j measure for diameter of ω_j
  - δ_j measure for overlap
  - $\gamma(\alpha)$ coarse space robustness indicator

$$\gamma(\alpha) = \max_j \delta_j^2 \left\| \alpha |\nabla \Psi_j|^2 \right\|_{L_\infty(\Omega)}$$

- guides choice of $\{\Psi_j\}$ w.r.t. α: energy minimisation

Energy Minimising Coarse Basis

- for given A and R_j
- coefficient vector of coarse basis function Ψ_j

$$r_j = R_j^T q_j$$

- energy minimisation problem

$$\min \sum_j q_j^T A_j q_j$$
$$\text{s.t.} \sum_j R_j^T q_j = \mathbb{1}$$

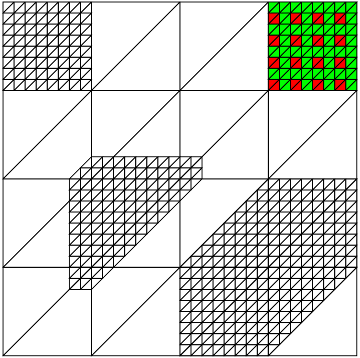
Construction of the Coarse Space

- basis functions defined locally $r_j = R_j^T q_j$
- solution of local problem $A_j q_j = g_j$
- well chosen right hand side g_j
- assume $g_j = R_j g \quad \Rightarrow \quad r_j = R_j^T A_j^{-1} R_j g$
- preservation of constants $\sum_j r_j = \mathbb{1}$

$$\sum_j r_j = \sum_j R_j^T A_j^{-1} R_j g = Bg = \mathbb{1}$$

- g : Lagrange multipliers of constrained minimisation problem

Example: Fine Scale Binary Medium



$\alpha = 10^6$

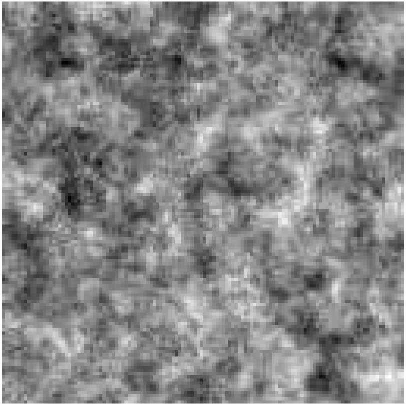
n_s	n	m	one	lin	linbc	oscbc	erg
4	8	32	24	34	34	24	25
8	8	64	40	59	62	27	27
16	8	128	77	112	115	26	26
32	8	256	154	219	240	26	26

$n_s = 32,\ n = 8,\ m = 256$

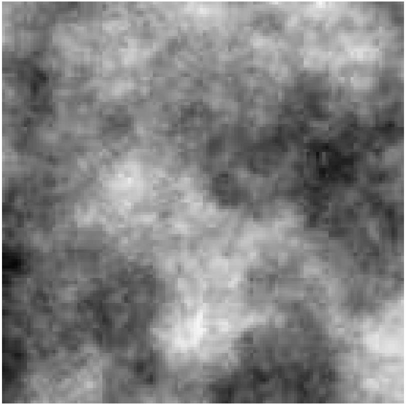
α	one	lin	linbc	oscbc	erg
10^0	129	22	22	22	22
10^2	132	81	52	23	23
10^4	132	218	218	25	26
10^6	154	219	240	26	26

Example: Gausian Random Field

mean 0, variance σ^2, correlation length λ



$\lambda = 5h$



$\lambda = 50h$

coefficient $\alpha = \exp \mathrm{Gaussian}$

Variance Robustness

σ^2	one	lin	linbc	oscbc	erg
0	67	22	22	22	23
2	162	44	40	36	35
4	226	65	55	46	44
8	377	121	94	65	62
12	531	199	146	86	81
16	662	304	213	108	103
20	819	440	297	133	126

Solving the Partition of Unity System

- how to solve the system $Bg = \mathbb{1}$?
- exploit structure of $B = \sum_i R_i^T A_i^{-1} R_i$

1. precondition B with A [Wan, Chan & Smith 2000]
   only as good as one-level method
2. diagonal preconditioner [Xu, Zikatanov 2004]
   $$D = \text{diag}(B)^{-1}$$
3. localised version of A
   $$E = \sum_i R_i^T A_i R_i$$
4. construct one-level preconditioner C for B

One-Level Preconditioner for a One-Level Preconditioner

- matrix A
- local problems for $A_i = R_i A R_i^T$
- one-level preconditioner $B = \sum_i R_i^T A_i^{-1} R_i$
- local problems $B_j = R_j B R_j^T$
- one-level preconditioner $C = \sum_j R_j^T B_j^{-1} R_j$
- A and A_i sparse, but B and B_j dense
- $B \sim A^{-1}$ and $C \sim B^{-1}$ so somehow $C \sim A$

Implementing the Partition of Unity Preconditioner

- consider a domain j with 2 neighbours k and l
- local problem solve

$$B_j^{-1} = (A_j^{-1} + \hat{I}_{jk} A_k^{-1} \hat{I}_{kj} + \hat{I}_{jl} A_l^{-1} \hat{I}_{lj})^{-1}$$

- Sherman-Morrison-Woodbury formula

$$B_j^{-1} = A_j - A_j \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix} H_{kl}^{-1} \begin{bmatrix} \hat{I}_{kj} \\ \hat{I}_{lj} \end{bmatrix} A_j$$

$$H_{kl} = \begin{bmatrix} A_k & \\ & A_l \end{bmatrix} + \begin{bmatrix} \hat{I}_{kj} \\ \hat{I}_{lj} \end{bmatrix} A_j \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix}$$

- sparse system solve

Robustness of the Partition of Unity Preconditioner

binary medium

α	A	D	E	C
10^0	53	43	18	10
10^2	70	108	56	10
10^4	71	119	134	9
10^6	71	37	200^+	9

n_s	n	m	A	D	E	C
4	8	32	18	33	167	10
8	8	64	31	37	200^+	10
16	8	128	52	38	200^+	10
32	8	256	71	37	200^+	9

Gaussian field

σ^2	A	D	E	C
0	38	44	18	10
2	96	94	37	13
4	138	164	55	14
8	200	200^+	96	15
12	200^+	200^+	152	16
16	200^+	200^+	199	16
20	200^+	200^+	200^+	17

Remarks

- coarse grid construction is **scalable**
- more costly than other coarsening strategies, but **“optimal”**
- **inaccurate** solve sufficient for $Bg = \mathbb{1}$
- important: choice of **supports**

  ideas: aggregation (strong connections), compatible relaxation, . . .

- alternative interpretation: optimal **BCs for multiscale FE**
- interesting also for numerical homogenisation (**upscaling**)

Summary

- considered elliptic equations with varying coefficients
- two-level preconditioner
  for a given set of overlapping supports
- construction is not cheap, but algebraic, scalable and robust
- main ideas
  - simple construction of coarse space
  - one-level preconditioner for one-level preconditioner
  - implementation via Sherman-Morrison-Woodbury
- topics for further research
  - convergence analysis
  - choice of supports
  - tensor coefficients
  - non-symmetric systems

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