Robust Coarsening for Domain Decomposition Methods

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Tuesday 26 June 2007

Problem

- elliptic equation

$$\nabla \cdot (\alpha \nabla u) = f$$

- highly variable coefficient α
- finite element discretisation
- system of equations

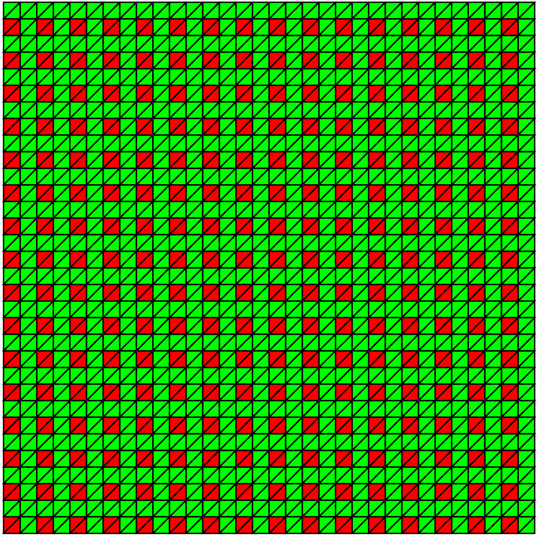
$$Au = f$$

Goals

- unstructured grids, varying coefficients:
  multilevel iterative methods
- domain decomposition:
  two-level additive Schwarz method
- scalable and robust methods:
  # iterations and cost per iteration well behaved w.r.t.
  - problem size, mesh resolution
  - number of subdomains
  - coefficients!

Example: Fine Scale Binary Medium

- problem size: $2^r \times 2^r$, $r = 5$
- coefficients: 1 green, α red



Scalability and Coefficient Robustness

iterations for two-level additive Schwarz as preconditioner for CG

$\alpha = 1$

r	one	lin	linbc	oscbc	erg
5	22	22	22	22	22
6	36	23	23	23	22
7	67	22	22	22	23
8	129	22	22	22	22

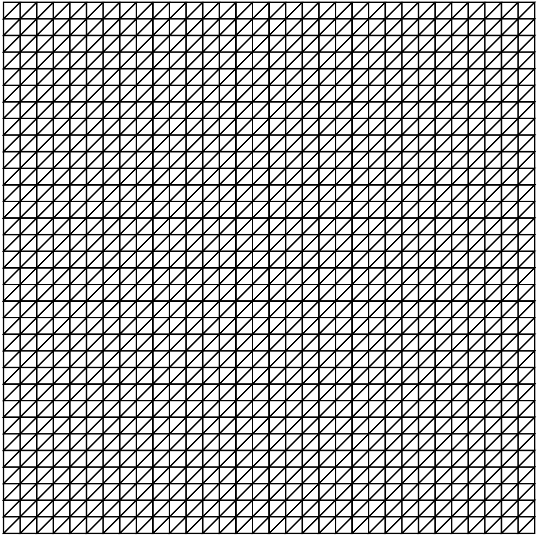
$\alpha = 10^6$

r	one	lin	linbc	oscbc	erg
5	24	34	34	24	25
6	40	59	62	27	27
7	77	112	115	26	26
8	154	219	240	26	26

Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

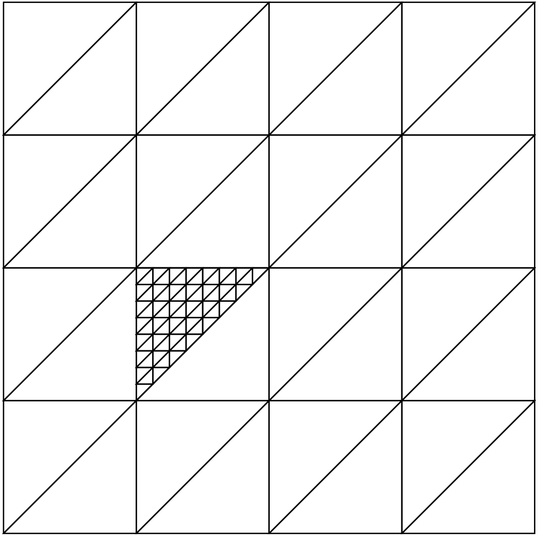
- fine grid



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

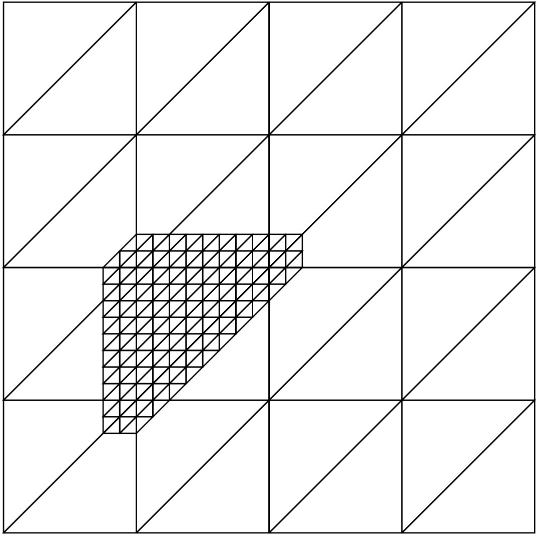
- fine grid
- subdomains



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

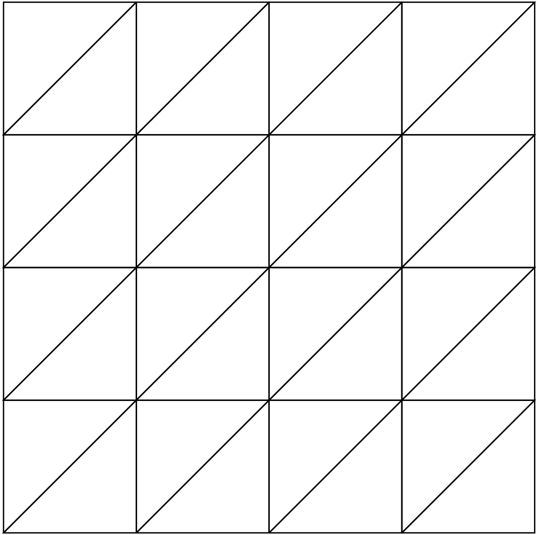
- fine grid
- subdomains
- overlap



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

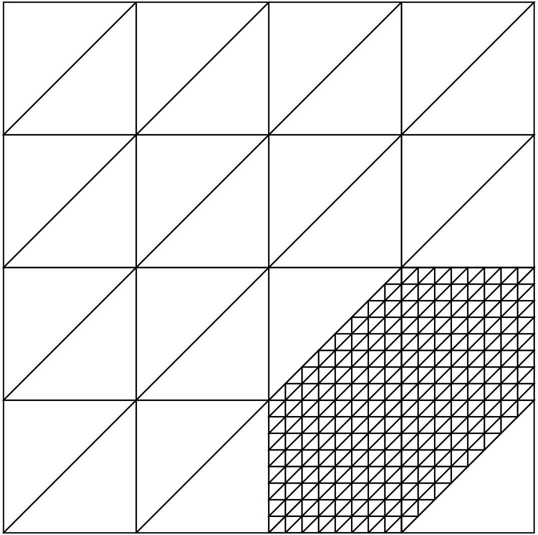
- fine grid
- subdomains
- overlap
- coarse grid



Domain Decomposition

Two-Level Overlapping Additive Schwarz Method

- fine grid
- subdomains
- overlap
- coarse grid
- coarse basis



Formulation of the Two-Level Method

- system $Au = f$
- restriction matrices R_i
- local problems $A_i = R_i A R_i^T$
- one-level preconditioner

$$B = \sum_i R_i^T A_i^{-1} R_i$$

- restriction matrix R_0
- coarse problem $A_0 = R_0 A R_0^T$
- two-level preconditioner

$$\tilde{B} = R_0^T A_0^{-1} R_0 + \sum_i R_i^T A_i^{-1} R_i$$

- columns r_i of R_0^T span coarse space

Construction of the Coarse Space (R_0)

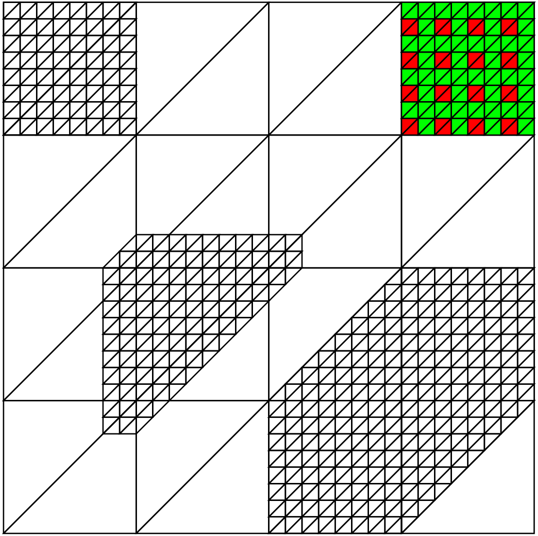
- basis functions defined locally $r_i = R_i^T q_i$
- solution of local problem $A_i q_i = g_i$
- well chosen right hand side g_i
- assume $g_i = R_i g \quad \Rightarrow \quad r_i = R_i^T A_i^{-1} R_i g$
- preservation of constants $\sum_i r_i = \mathbf{1}$

$$\sum_i r_i = \sum_i R_i^T A_i^{-1} R_i g = Bg = \mathbf{1}$$

- g : Lagrange multipliers of constrained minimisation problem

Example: Fine Scale Binary Medium

- coarse grid: 4×4
- fine grid: 8×8
- problem size: $2^r \times 2^r$, $r = 5$
- coefficients: 1 green, α red
- subdomains
- coarse basis



Scalability and Coefficient Robustness

$\alpha = 10^6$

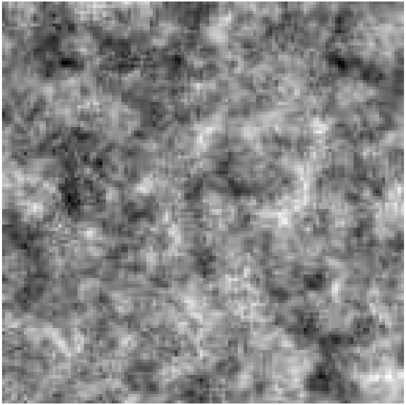
r	one	lin	linbc	oscbc	erg
5	24	34	34	24	25
6	40	59	62	27	27
7	77	112	115	26	26
8	154	219	240	26	26

$r = 8$

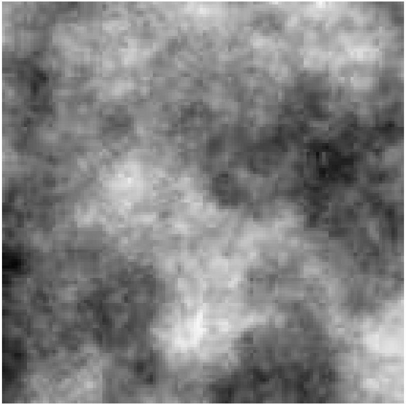
α	one	lin	linbc	oscbc	erg
10^0	129	22	22	22	22
10^2	132	81	52	23	23
10^4	132	218	218	25	26
10^6	154	219	240	26	26

Example: Gaussian Random Field

mean 0, variance σ^2, correlation length λ



$\lambda = 5h$



$\lambda = 50h$

coefficient $\alpha = \exp \mathrm{Gaussian}$

Variance Robustness

σ^2	one	lin	linbc	oscbc	erg
0	67	22	22	22	23
2	162	44	40	36	35
4	226	65	55	46	44
8	377	121	94	65	62
12	531	199	146	86	81
16	662	304	213	108	103
20	819	440	297	133	126

Preconditioning the One-Level Preconditioner

- how to solve the system $Bg = \mathbf{1}$?
- precondition B with A
  only as good as one-level method
- B has special structure, “local” operator
- no global solve needed
- construct one-level preconditioner C for B
- other ideas
  - diagonal preconditioner : $D = \text{diag}(B)^{-1}$
  - localised version of A : $E = \sum_i R_i^T A_i R_i$

One-Level Preconditioner for the One-Level Preconditioner

- matrix A
- local problems for $A_i = R_i A R_i^T$
- one-level preconditioner $B = \sum_i R_i^T A_i^{-1} R_i$
- local problems $B_j = R_j B R_j^T$
- one-level preconditioner $C = \sum_j R_j^T B_j^{-1} R_j$
- A and A_i sparse, but B and B_j dense
- $B \sim A^{-1}$ and $C \sim B^{-1}$ so somehow $C \sim A$

Implementing the Preconditioner

- consider a domain j with 2 neighbours k and l
- local problem solve

$$B_j^{-1} = (A_j^{-1} + \hat{I}_{jk} A_k^{-1} \hat{I}_{kj} + \hat{I}_{jl} A_l^{-1} \hat{I}_{lj})^{-1}$$

- Sherman-Morisson-Woodbury formula

$$B_j^{-1} = A_j - A_j \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix} H_{kl}^{-1} \begin{bmatrix} \hat{I}_{kj} \\ \hat{I}_{lj} \end{bmatrix} A_j$$

$$H_{kl} = \begin{bmatrix} A_k & \\ & A_l \end{bmatrix} + \begin{bmatrix} \hat{I}_{kj} \\ \hat{I}_{lj} \end{bmatrix} A_j \begin{bmatrix} \hat{I}_{jk} & \hat{I}_{jl} \end{bmatrix}$$

- sparse system solve

Robustness of Coarse Space Construction

binary medium

α	mat	diag	loc	as
10^0	53	43	18	10
10^2	70	108	56	10
10^4	71	119	134	9
10^6	71	37	200^+	9

r	mat	diag	loc	as
5	18	33	167	10
6	31	37	200^+	10
7	52	38	200^+	10
8	71	37	200^+	9

Gaussian field

σ^2	mat	diag	loc	as
0	38	44	18	10
2	96	94	37	13
4	138	164	55	14
8	200	200^+	96	15
12	200^+	200^+	152	16
16	200^+	200^+	199	16
20	200^+	200^+	200^+	17

Summary

- considered elliptic equations with varying coefficients
- two-level preconditioner
  for a given set of overlapping subdomains
- construction is not cheap, but algebraic, scalable and robust
- main ideas
  - simple construction of coarse space
  - one-level preconditioner for one-level preconditioner
  - linear algebra trick
- topics for further research
  - convergence analysis
  - choice of subdomains
  - non-symmetric systems

References

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